Revision Notes
Chapter - 1
SETS

- **Set**: A set is a well-defined collection of objects.
- **Representaiton of sets**: (i) Roster or Tabular form, (ii) Rule method or set builder form.

Types of sets:

- **Empty set**: A set which does not contain any element is called empty set or null set or void set. It is denoted by $\phi\phi$ or { }.
- **Singleton set**: A set, consisting of a single element, is called a singleton set.
- **Finite set**: A set which consists of a definite number of elements is called finite set.
- **Infinite set**: A set, which is not finite, is called infinite set.
- **Equivalent sets**: Two finite sets A and B are equivalent, if their cardinal numbers are same, .i.e, $n(A)=n(B)n(A)=n(B)$.
- **Equal sets**: Two sets A and B are said to be equal if they have exactly the same elements.
- **Subset**: A set A is said to be subset of a set B, if every element of A is also an element of B. Intervals are subsets of R.
- **Proper set**: If $A \subseteq\subseteq B$ and $A \neq\neq B$, then A is called a proper set of B, written as $A \subset\subset B$.
- **Universal set**: If all the sets under consideration are subsets of a large set U, then U is known as a universal set. And it is denoted by rectangle in Venn-Diagram.
- **Power set**: A power set of a set A is collection of all subsets of A. It is denoted by P(A).
- **Venn-Diagram**: A gepmetrical figure illustrating universal set, subsets and their operations is known as Venn-Diagram.
- **Union of sets**: The union of two sets A and B is the set of all those elements which are either in A or in B.
- **Intersection of sets**: The intersection of two sets A and B is the set of all elements which are common. The difference of two sets A and B in this order is the set of elements which belong to A but not to B.
- **Disjoint sets**: Two sets A and B are said to be disjoint, if $A\cap B=\phi A\cap B=\phi$.
- **Difference of sets**: Difference of two sets i.e., set (A - B) is the set of those elements of A which do not belong to B.
- **Compliment of a set:** The complement of a subset A of universal set U is the set of all elements of U which are not the elements of A. $A' = U - A$.
- For any two sets A and B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$
- If A and B are finite sets such that $A \cap B = \varphi$, then
  $n (A \cup B) = n (A) + n (B)$.
- If $A \cap B \neq \varphi$, then
  $n (A \cup B) = n (A) + n (B) – n (A \cap B)$

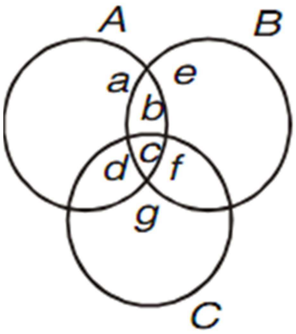
Q1. In a survey it is found that 21 people like product A, 26 people like product B and 29 like product C. If 14 people like product A and B, 15 people like product B and C, 12 people like product C and A, and 8 people like all the three products. Find

(i) How many people are surveyed in all?

(ii) How many like product C only?

Ans. Hint : Let A, B, C denote respectively the set of people who like product

A, B, C.



a, b, c, d, e, f, g – Number of elements in bounded region

(i) Total number of Surveyed people = a + b + c + d + e + f + g = 43

(ii) Number of people who like product C only = g = 10

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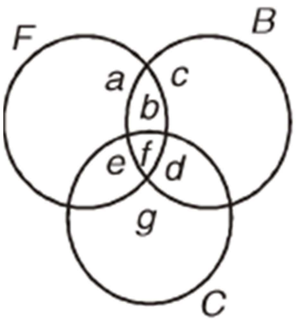
2. A college awarded 38 medals in football, 15 in basket ball and 20 in cricket. If these medals went to a total of 50 men and only five men got medals in all the three sports, how many received medals in exactly two of the three sports?

Ans. people got medals in exactly two of the three sports.

Hint :



f = 5

a + b + f + e = 38

b + c + d + f = 15

e + d + f + g = 20

a + b + c + d + e + f + g = 50

we have to find b + d + e

3.There are 200 individuals with a skin disorder, 120 had been exposed to the chemical C_1, 50 to chemical C_2, and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to

(1) chemical C_1 but not chemical C_2

(2) chemical C_2 but not chemical C_1

(4) chemical C_1 or chemical C_2

Ans. A denote the set of individuals exposed to the chemical C_1 and B denote the set of individuals exposed to the chemical C_2

$n(U) = 200$, $n(A) = 120$, $n(B) = 50$, $n(A \cap B) = 30$

(i) $n(A-B) = n(A) - n(A \cap B)$

$=120-30=90$

(ii)$n(B-A) = n(B) - n(A \cap B)$

$=50-30 = 20$

(iii)$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$=120+50-30$

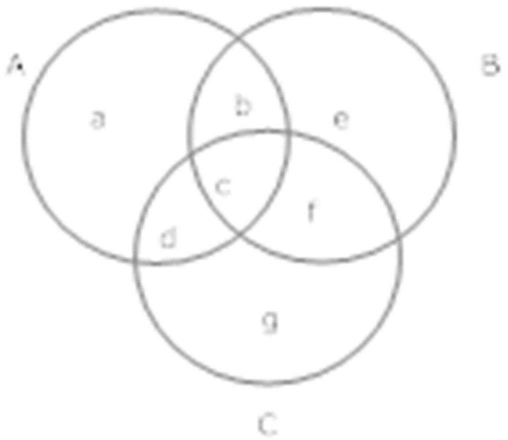
$=140$

4.In a survey it was found that 21 peoples liked product A, 26 liked product B and 29 liked product C. If 14 people liked products A and B, 12 people like C and A, 15 people like B and C and 8 liked all the three products. Find now many liked product C only.

Ans. a + b + c + d = 21

b + c + e + f = 26

c + d + f + g = 29



$b + c = 14, c + f = 15, c + d = 12$

$c = 8$

$d = 4, c = 8, f = 7, b = 6, g = 10, e = 5, a = 3$

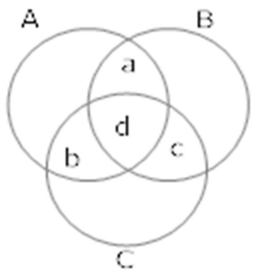
like product c only = g = 10

5.A college awarded 38 medals in football, 15 in basketball and 20 in cricket. If these medals went to a total of 58 men and only three men got medal in all the three sports, how many received medals in exactly two of the three sports?

Ans. Let A, B and C denotes the set of men who received medals in football, basketball and cricket respectively.

$n(A) = 38,\ n(B) = 15,\ n(C) = 20$

$n(A \cup B \cup C) = 58$ and $n(A \cap B \cap C) = 3$



$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$58 = 38 + 15 + 20 - (a + d) - (d + c) - (b + d) + 3$

$18 = a + d + c + b + d$

$18 = a + b + c + 3d$

$18 = a + b + c + 3 \times 3$

$9 = a + b + c$

6.In a survey of 60 people, it was found that 25 people read news paper H, 26 read newspaper T,

26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three

newspaper. Find

(i) The no. of people who read at least one of the newspapers.

(ii) The no. of people who read exactly one news paper.

Ans. $a + b + c + d = 25$

$b + c + e + f = 26$

c + d + f +g = 26

c + d = 9

b + c = 11

c + f = 8

c = 3

f = 5, b = 8, d = 6, c = 3, g = 12

e = 10, a = 8

(i) a + b + c + d + e + f + g = 52

(ii) a + e + g = 30

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7. In a survey of 100 students, the no. of students studying the various languages were found to be English only 18, English but not Hindi 23, English and Sanskrit 8, English 26, Sanskrit 48, Sanskrit and Hindi 8, no language 24. Find

(i) How many students were studying Hindi?

(ii) How many students were studying English and Hindi?

Ans. $\cup = 100, a = 18$

$$a + e = 23, e + g = 8$$

$$a + e + g + d = 26$$

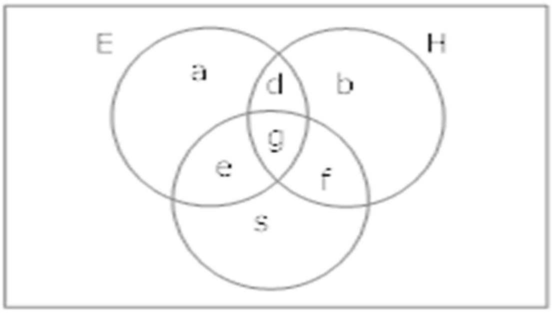
$$e + g + f + c = 48$$

$$g + f = 8$$

$$so, e = 5, g = 3, d = 0, f = 5, c = 35$$

$$(i)\ d + g + f + b = 0 + 3 + 5 + 10 = 18$$

$$(ii)\ d + g = 0 + 3 = 3$$



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8. In a class of 50 students, 30 students like Hindi, 25 like science and 16 like both.

Find the no. of students who like

(i) Either Hindi or science

(ii) Neither Hindi nor science.

Ans. Let ∪ = all the students of the class ,H = students who like Hindi

S = Students who like Science

(i) $n(H \cup S) = n(H) + n(S) - n(H \cap S)$

$= 30 + 25 - 16$

$= 39$

(ii) $n(H' \cap S') = n(H \cup S)'$

$= \cup - n(H \cup S)$

$= 50 - 39$

$= 11$

9.In a town of 10,000 families, it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C. 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three papers. Find the no. of families which buy

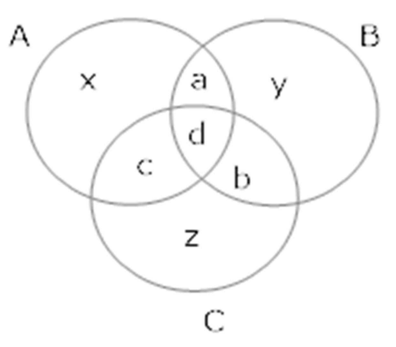
(i) A only (ii) B only (iii) none of A, B, and C.

Ans. $x + a + c + d = 4000$

$y + a + d + b = 2000$

$z + b + c + d = 1000$

$a + d = 500$, $b + d = 300$, $C + d = 400$ $d = 200$



On Solving $a = 300$, $b = 100$, $c = 200$

(i) $x = 4000 - 300 - 200 - 200 = 3300$

(ii) $y = 2000 - 300 - 200 - 100 = 1400$

(iii) $z = 1000 – 100 – 200 – 200 = 500$

None of these = 10,000 – (3300 + 1400 + 500 + 300+ 100 + 200 + 200)

= 10,000 – 6000

= 4000

10.Two finite sets have m and n elements. The total no. of subsets of the first set is 56 more than the total no. of subsets of second set. Find the value of m and n.

Ans. Let A and B be two sets having m and n elements respectively

no of subsets of A = 2^m

no of subsets of B = 2^n

According to question

$2^m = 56 + 2^n$

$2^m - 2^n = 56$

$2^n (2^{m-n} – 1) = 56$

$2^n (2^{m-n} – 1) = 2^3 (2^3 – 1)$

$2^n = 2^3$

$n = 3$

$m – n = 3$

$m – 3 = 3$

$m = 6$

UNIVERSITY QUESTION UNIT VI

2015

13. (a) Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x + y \le 1$. 7½

(b) Find the volume under the plane $x+y+z=6$ and above the triangle in the xy-plane bounded by $2x=3y$, $y=0$, $x=3$. 7½

2016

5. Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\,dx\,dy$.

12. (a) Find the area between the line $y = x$ and curve $y = x^2$ enclosed in first quadrant.

(b) Evaluate by changing the order of integration :

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}.$$

2017

7. Evaluate $\iint r^3 dr d\theta$ over the area bounded between the circles $r=2\cos\theta$ & $r=4\cos\theta$

13. (i) Evaluate the double integral

$$\int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2 \, dxdy$$

(ii) Evaluate the triple integral

$\iiint(x^2+y^2+z^2)\, dx\, dy\, dz$ where R denotes the region bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=a$, $a>0$

2018

5. Evaluate $\int_0^{\pi/2} \int_0^{\sin\theta} r d\theta dr.$ 3

10. Change the order of integration : 15

$$\int_0^a \int_x^{a^2/x} \phi(x, y) dxdy.$$

11. (a) Evaluate $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 dxdydz.$ 7½

(b) Evaluate $\int_0^1 \int_0^{x^2} e^{y/x} dxdy.$ 7½

2019

5. Evaluate the triple integral $\int_0^1 \int_1^2 \int_2^3 dx\, dy\, dz$.

7. Find the area of the region bounded by the circle $x^2 + y^2 = a^2$, by double integration.

13. (i) Evaluate the double integral $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} x^2 y\, dx\, dy$. Also mention the region of integration involved in this double integral.

(ii) Prove that the value of triple integration : $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\, dz\, dy\, dx$, is $\frac{1}{48}$.

UNIT- VI

{UNIVERSITY QUESTION PAPERS}

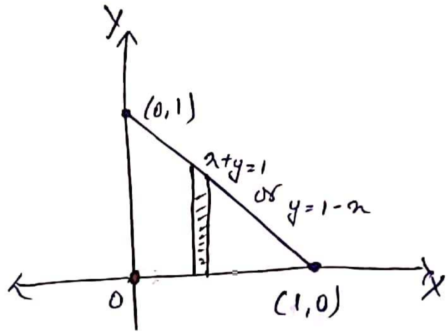
2015

Q13 (a) Evaluate : $\iint xy\,dx\,dy$ over the region in the +ve Quad. for which $x+y \le 1$.

Soln: For point, we put $x=0$ in eqn $\boxed{x+y=1}$

$y=0$; $y=1$

$x=0$; $x=1$ $\therefore$ Points are $(0,1)$ $(1,0)$



To cover this region,

Limits are ; $0 \le y \le 1-x$

$0 \le x \le 1$

$$\iint_R x\,dx\,dy = \int_{x=0}^{1} x\left[\int_{y=0}^{1} y\,dy\right]dx$$

$$\Rightarrow \int_0^1 x\left[\frac{y^2}{2}\right]_0^1 dx$$

$$\Rightarrow \int_0^1 x\,\frac{(1-x)^2}{2}\,dx$$

$$\Rightarrow \int_0^1 x\,\frac{(1-2x+x^2)}{2}\,dx$$

$$\Rightarrow \frac{1}{2}\int_0^1 (x-2x^2+x^3)\,dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^1$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$$

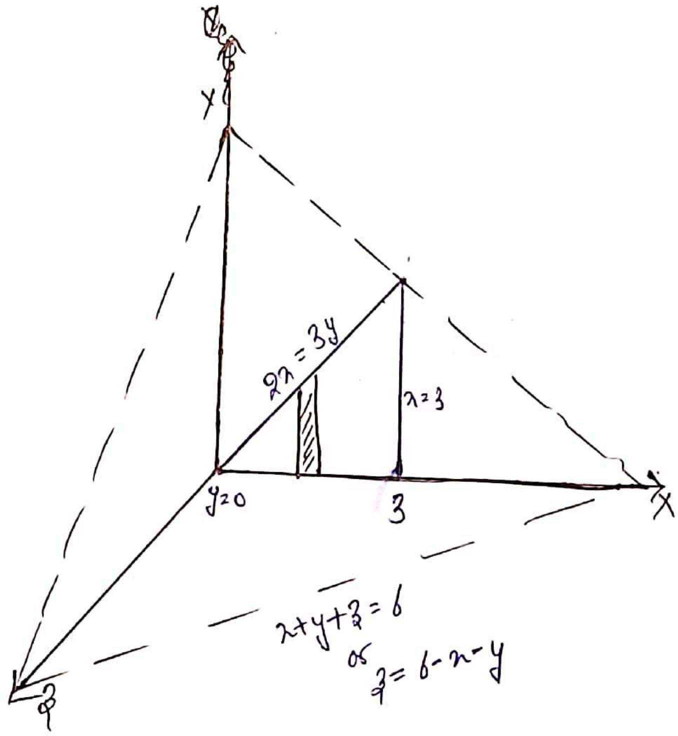
$$\Rightarrow \frac{1}{2}\left[\frac{1}{12}\right]$$

$$\Rightarrow \boxed{\frac{1}{24}\text{ Sq. Unit}}$$ Ans

(b) Find the Volume under the plane $x+y+z=6$ & above the triangle in the xy-plane bounded by $2x=3y$, $y=0$, $x=3$.

Soln:- Limits are-

$$0 \le x \le 3$$
$$0 \le y \le \frac{2}{3}x$$
$$0 \le z \le 6-x-y$$



$$V = \iiint dx\,dy\,dz$$

$$V = \int_{x=0}^{3}\int_{y=0}^{\frac{2}{3}x}\int_{z=0}^{6-x-y} dz\,dy\,dx$$

$$V = \int_0^3\int_0^{\frac{2}{3}x} [z]_0^{6-x-y}\,dy\,dx$$

$$V = \int_0^3\int_0^{2/3x} (6-x-y)\,dy\,dx$$

$$V = \int_0^3 \left[6y - xy - \frac{y^2}{2}\right]_0^{\frac{2}{3}x} dx$$

$$V = \int_0^3 6\times\frac{2}{3}x - x\times\frac{2}{3}x - \left(\left(\frac{2}{3}x\right)^2\times\frac{1}{2}\right) dx$$

$$V = \int_0^3 4x - \frac{2x^2}{3} - \frac{2x^2}{9}\,dx$$

$$V = \left[\frac{4x^2}{2} - \frac{2x^3}{9} - \frac{2x^3}{27}\right]_0^3$$

$$V = \left[\frac{4\times 9}{2} - \frac{2\times 27}{9} - \frac{2\times 27}{27}\right]$$

$$V = (18-6-2)$$

$\boxed{V = 10 \text{ cubit units}}$ Ans

2016

Q5 Evaluate :- $\int_0^3 \int_1^2 xy(1+x+y)\,dx\,dy$

Soln: Let $I = \int_0^3 \int_1^2 (xy + x^2y + xy^2)\,dx\,dy$

$I = \int_0^3 \left[\frac{x^2}{2}y + \frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_1^2 dy$

$I = \int_0^3 \left[\left[\frac{4}{2}y + \frac{8}{3}y + \frac{4}{2}y^2\right] - \left[\frac{1}{2}y + \frac{1}{3}y + \frac{1}{2}y^2\right]\right] dy$

$I = \int_0^3 \left[\frac{3}{2}y + \frac{7}{3}y + \frac{3}{2}y^2\right] dy$

$I = \int_0^3 \left[\frac{23}{6}y + \frac{3}{2}y^2\right] dy$

$I = \left[\frac{23}{6}\frac{y^2}{2} + \frac{3}{2}\cdot\frac{y^3}{3}\right]_0^3$

$I = \left[\frac{23}{6} \times \frac{9}{2} + \frac{3}{2} \times \frac{27}{3}\right]$

$I = \frac{69}{4} + \frac{27}{2} \Rightarrow \boxed{\frac{153}{4}}$ Ans

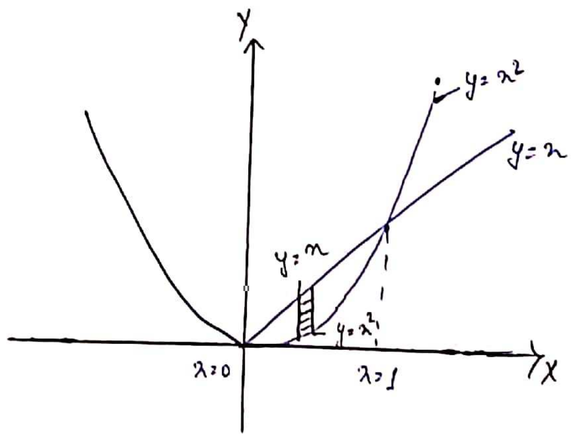
Q12 (a) Find the area b/w the line $y = x$ & curve $y = x^2$ enclosed in first Quadrant.

Soln:- For Points: $y = x$; $y = x^2$

$x = x^2 \Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$\therefore \boxed{x=0}$; $\boxed{x=1}$



Limits are $0 \le x \le 1$

$x^2 \le y \le x$

$\Rightarrow \iint_R dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{x} 1\cdot dy\,dx$

$\Rightarrow \int_0^1 [y]_{x^2}^{x}\,dx$

$\Rightarrow \int_0^1 (x - x^2)\,dx$

$\Rightarrow \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

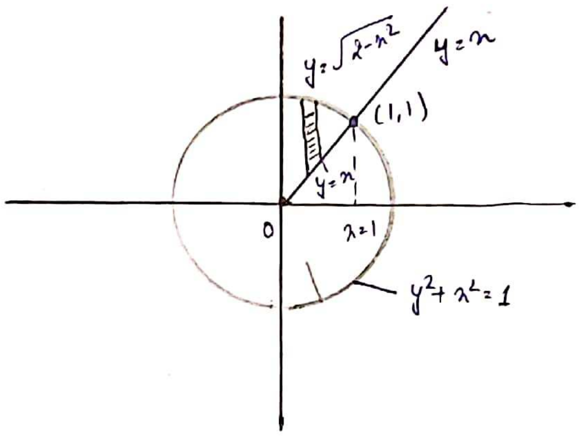
$\Rightarrow \left(\frac{1}{2} - \frac{1}{3}\right) \Rightarrow \boxed{\frac{1}{6} \text{ Sq. Unit}}$ Ans

Q12 (b) Evaluate by changing the Order of integration :-

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

Soln :- First we have to know the Order :-

$$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$



Limits are :- $x \le y \le \sqrt{2-x^2}$

$0 \le x \le 1$

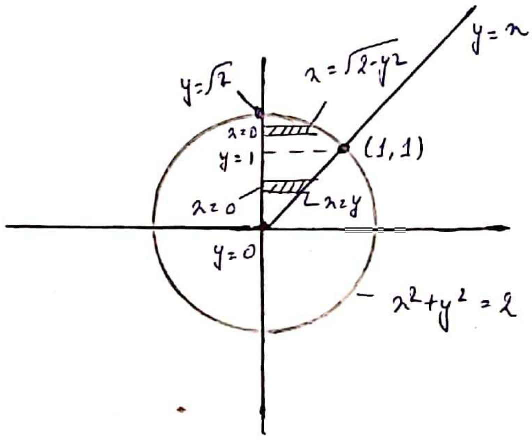
From here we get,

$\boxed{x=y}$ & $\boxed{y=\sqrt{2-x^2}}$

$\Rightarrow y = \sqrt{2-x^2}$

Sq. both the sides

$\boxed{x^2+y^2=2}$ - eqn of circle



After Change the Order of Integration ;

$0 \le y \le 1$ & $1 \le y \le \sqrt{2}$

$0 \le x \le y$ & $0 \le x \le \sqrt{2-y^2}$

$$I = \int_0^1 \int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \iint_R dx\,dy = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$\Rightarrow \int \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ Let $x^2+y^2=t$

$2x\,dx = dt \Rightarrow \boxed{x\,dx = \frac{1}{2}dt} \Rightarrow \frac{1}{2}\int dt \Rightarrow \frac{1}{2}(2\sqrt{t}) \Rightarrow \sqrt{t} \Rightarrow \sqrt{x^2+y^2}$

$$\Rightarrow I = \int_0^1 \left(\sqrt{x^2+y^2}\right)_0^y dy + \int_1^{\sqrt{2}} \left(\sqrt{x^2+y^2}\right)_0^{\sqrt{2-y^2}} dy$$

$$I = \int_0^1 \sqrt{y^2+y^2} - \sqrt{y^2} + \int_1^{\sqrt{2}} \sqrt{2-y^2+y^2} - \sqrt{y^2}\,dy$$

$$I = \int_0^1 \sqrt{2}y - y \cdot dy + \int_1^{\sqrt{2}} \sqrt{2} - y\,dy$$

$$I = \left[\sqrt{2}\cdot\frac{y^2}{2} - \frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2}\right]_0^{\sqrt{2}}$$

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$$I = \left(\sqrt{2}\cdot\frac{1}{2} - \frac{1}{2}\right) + \left(\sqrt{2}\cdot\sqrt{2} - \frac{(\sqrt{2})^2}{2}\right) - \left(\sqrt{2} - \frac{1}{2}\right)$$

$$I = \left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right) + \left(2 - \frac{2}{2}\right) - \left(\frac{2\sqrt{2}-1}{2}\right)$$

$$I = \frac{\sqrt{2}-1}{2} + 1 - \frac{2\sqrt{2}-1}{2}$$

$$I = \frac{\sqrt{2}-1+1}{2} - \frac{2\sqrt{2}-1}{2} \Rightarrow \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}-1}{2}$$

$$\boxed{I \Rightarrow \frac{2\sqrt{2}-1}{2}} \text{ Ans}$$

— x — x — x — x — x — x — x —

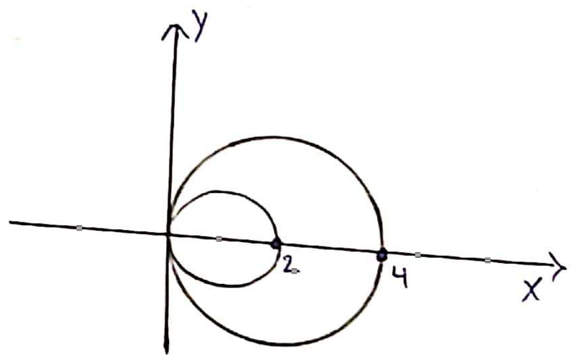
2017

Q7 Evaluate $\iint r^3\,dr\,d\theta$ over the area bounded b/w the circles $r = 2\cos\theta$ & $r = 4\cos\theta$.

Sol^n:- $r = 2\cos\theta$; $r = 4\cos\theta$

From eq^n of circle, $r = 0$ when

$\cos\theta = 0 \Rightarrow \theta = \pm \pi/2$



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; $2\cos\theta \le r \le 4\cos\theta$

$$\therefore \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta \Rightarrow \int_{-\pi/2}^{\pi/2}\left[\frac{64\cdot 4\cos^4\theta}{4} - \frac{16\cos^4\theta}{4}\right]d\theta$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2}\left(64\cos^4\theta - 4\cos^4\theta\right)d\theta \Rightarrow \int_{-\pi/2}^{\pi/2} 60\cos^4\theta\,d\theta$$

$$\Rightarrow 60\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta \Rightarrow 60\times 2\int_0^{\pi/2}\cos^4\theta\,d\theta \quad \{\text{as } \cos\theta \text{ is an even } f^n,\ \cos(-\theta) = \cos\theta\}$$

$$\Rightarrow 120\int_0^{\pi/2}\cos^4\theta\,d\theta.$$

Rough

$\cos^4\theta = (\cos^2\theta)^2$

$= \left(\frac{\cos 2\theta + 1}{2}\right)^2$

$= \frac{1}{4}\left(\cos^2 2\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\right)$

$= \frac{1}{4}\left(\cos\frac{4\theta}{2} + 1\right) + \frac{1}{2}\cos 2\theta + \frac{1}{4}$

$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4} + \frac{1}{8}$

$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

Now, $120\int_0^{\pi/2}\left(\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}\right)d\theta$

$$\Rightarrow 120\left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8}\right]_0^{\pi/2}$$

$$\Rightarrow 120\left[\frac{\sin 2\pi}{32} + \frac{\sin\pi}{4} + \frac{3}{8}\left(\frac{\pi}{2}\right)\right]$$

$$\Rightarrow 120\left[0 + 0 + \frac{3\pi}{16}\right]$$

$$\Rightarrow \overset{45}{\cancel{120}}\left(\frac{3\pi}{\cancel{16}_{2}}\right) \Rightarrow \boxed{\frac{45\pi}{2}} \text{ Ans}$$

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Q13 (a) Evaluate the double integral :-

$$\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\,dx\,dy$$

Soln :- $$I = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\,dx\,dy$$

$$I = \int_{-a}^{a} 2\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy + 0 \quad \{2xy \text{ being an odd } f^n \text{ of } y, \text{ it's int. is zero}\}$$

$$I = 2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$I = 2\int_{-a}^{a}\left\{x^2\cdot\frac{b\sqrt{a^2-x^2}}{a}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right\}dx$$

$$I = 2\cdot 2\int_{0}^{a}\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\,dx \quad \left\{\int_{-a}^{a}f(x)\,dx = 2\int_0^a f(x)\,dx\right\}$$

$$I = \frac{4b}{a}\int_{0}^{a} x^2\sqrt{a^2-x^2}+\frac{b^2}{3a^2}(a^2-x^2)^{3/2}\,dx$$

Put $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$.

$$I = \frac{4b}{a}\int_{0}^{\pi/2}\left\{a^2\sin^2\theta\cos\theta+\frac{b^2}{3a^2}a^3\cos^3\theta\right\}a\cos\theta\,d\theta$$

$$I = \frac{4b}{a}\cdot a^2\int_{0}^{\pi/2}\left(a^2\sin^2\theta\cos^2\theta+\frac{b^2}{3}\cos^4\theta\right)d\theta$$

$$I = 4ab\left[a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta+\frac{b^2}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta\right]$$

$$I = 4ab\left[a^2\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{b^2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$I = 4ab\left[\frac{\pi a^2}{16}+\frac{\pi b^2}{16}\right]$$

$$\boxed{I = \frac{1}{4}\pi ab\,(a^2+b^2)}$$ Ans

Q13 (b) Evaluate: $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ where R denotes the region bounded by $x=0,\ y=0,\ z=0$ & $x+y+z=a,\ a>0$.

Soln:-
$$\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2+y^2+z^2)\,dz\,dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{a-x} \left[(x^2+y^2)z + \frac{z^3}{3}\right]_0^{a-x-y} dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{a-x} \left\{(x^2+y^2)\left[(a-x)-y\right] + \frac{\left[(a-x)-y\right]^3}{3}\right\} dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{a-x} \left\{x^2(a-x) - x^2y + y^2(a-x) - y^3 + \frac{(a-x)^3}{3} - \frac{y^3}{3} - \frac{1}{3}\cdot 3(a-x)^2 y + \frac{1}{3}\cdot 3(a-x)y^2\right\} dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{a-x} \left\{x^2(a-x) + \frac{1}{3}(a-x)^3 + y^2(a-x) - y^3 - \frac{y^3}{3} - x^2y - (a-x)^2\cdot y + (a-x)\cdot y^2\right\} dy\,dx$$

$$\Rightarrow \int_0^a \int_0^{a-x} \left\{x^2(a-x) + \frac{1}{3}(a-x)^3 + 2y^2(a-x) - \frac{4}{3}y^3 - x^2y - (a-x)^2 y\right\} dy\,dx$$

$$\Rightarrow \int_0^a \left[x^2(a-x)y + \frac{1}{3}(a-x)^3 y + 2(a-x)\frac{y^3}{3} - \frac{4}{3}\cdot\frac{y^4}{4} - x^2\cdot\frac{y^2}{2} - (a-x)^2\cdot\frac{y^2}{2}\right]_0^{a-x} dx$$

$$\Rightarrow \int_0^a \left\{x^2(a-x)^2 + \frac{1}{3}(a-x^4) + \frac{2}{3}(a-x)^4 - \frac{1}{3}(a-x)^4 - \frac{1}{2}x^2(a-x)^2 - \frac{(a-x)^4}{2}\right\} dx$$

$$\Rightarrow \int_0^a \left\{\frac{1}{2}x^2(a-x)^2 + \frac{1}{6}(a-x)^4\right\} dx$$

$$\Rightarrow \int_0^a \frac{(a-x)^2}{2}\left\{x^2 + \frac{1}{3}(a-x)^2\right\} dx$$

$$\Rightarrow \int_0^a \frac{(a-x)^2}{2}\left\{\frac{3x^2+a^2+x^2-2ax}{3}\right\} dx$$

$$\Rightarrow \frac{1}{6}\int_0^a \left\{(a^2+x^2-2ax)(a^2+4x^2-2ax)\right\} dx$$

$$\Rightarrow \frac{1}{6}\int_0^a \left\{a^4 + 4a^2x^2 - 2a^3x + a^2x^2 + 4x^4 - 2ax^3 - 2a^3x - 8ax^3 + 4a^2x^2\right\} dx$$

$$\Rightarrow \int_0^a \left\{a^4 + 9a^2x^2 + 4x^4 - 4a^3x - 10ax^3\right\} dx$$

$$\Rightarrow \left[a^4x + 9a^2\frac{x^3}{3} + \frac{4x^5}{5} - \frac{4a^3x^2}{2} - 10\frac{ax^4}{4}\right]_0^a$$

$\Rightarrow a^5 + \frac{9a^5}{3} + \frac{4a^5}{5} - \frac{4a^5}{2} - \frac{10a^5}{4}$

$\Rightarrow a^5 + 3a^5 + \frac{4}{5}a^5 - 2a^5 - \frac{5}{2}a^5$

$\Rightarrow 2a^5 + \frac{4}{5}a^5 - \frac{5}{2}a^5$

$\Rightarrow 2a^5 + \frac{8a^5 - 25a^5}{10}$

$\Rightarrow \frac{20a^5 - 17a^5}{10} \Rightarrow \boxed{\frac{3}{10}a^5}$ Ans

—x — x — x — x — x — x — x —

2018

Q5 Evaluate: $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr.$

Soln: Let $I = \int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$

$I = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$

$I = \frac{1}{2}\int_0^{\pi/2} [\sin^2\theta - 0]\,d\theta$

$I = \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta$

$I = \frac{1}{2}\int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$

$I = \frac{1}{4}\int_0^{\pi/2} (1-\cos 2\theta)\,d\theta$

$I = \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$I = \frac{1}{4}\left[\frac{\pi}{2} - \frac{\sin\pi}{2}\right]$

$I = \frac{1}{4}\left(\frac{\pi}{2}\right) \Rightarrow \boxed{I = \frac{\pi}{8}}$ Ans

Q11 (a) Evaluate: $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 \, dx \, dy \, dz$

Soln: $I = \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 \cdot 1 \, dz \, dy \, dx$

$I = \int_0^a \int_0^{a-x} [z]_0^{a-x-y} x^2 \, dy \, dx$

$I = \int_0^a \int_0^{a-x} (a-x-y) x^2 \, dy \, dx$

$I = \int_0^a \left[ay - xy - \frac{y^2}{2}\right]_0^{a-x} \cdot x^2 \, dx$

$I = \int_0^a \left(a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right) x^2 \, dx$

$I = \frac{1}{2} \int_0^a \left(2a^2 - \cancel{2ax} - 2ax + 2x^2 - a^2 + x^2 + \cancel{2ax}\right) x^2 \, dx$

$I = \frac{1}{2} \int_0^a (a^2 - 2ax + x^2) x^2 \, dx$

$I = \frac{1}{2} \int_0^a a^2 x^2 - 2a x^3 + x^4 \, dx$

$I = \frac{1}{2} \left[a^2 \frac{x^3}{3} - 2a \frac{x^4}{4} + \frac{x^5}{5}\right]_0^a$

$I = \frac{1}{2} \left[a^2 \cdot \frac{a^3}{3} - 2a \cdot \frac{a^4}{4} + \frac{a^5}{5}\right]$

$I = \frac{1}{2} \left[\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5}\right]$

$I = \frac{a^5}{2} \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right]$

$I = \frac{a^5}{2} \left[\frac{10 - 15 + 6}{30}\right] \Rightarrow \frac{a^5}{16}$ Ans

Q11 (b) Evaluate :- $\int_0^1 \int_0^{x^2} e^{y/x}\, dx\, dy$

Soln :- $I = \int_0^1 \int_0^{x^2} 1 \cdot e^{y/x}\, dy\, dx$

$I = \int_0^1 \left[x \cdot e^{y/x} \right]_0^{x^2} dx$

$I = \int_0^1 x \cdot e^{x} - x\, dx$

$I = \int_0^1 x \cdot e^{x}\, dx - \int_0^1 x\, dx$

$I = \left[xe^{x} - e^{x} \right]_0^1 - \left[\frac{x^2}{2} \right]_0^1$

$I = e^1 - e^1 + 1 - \frac{1}{2}$

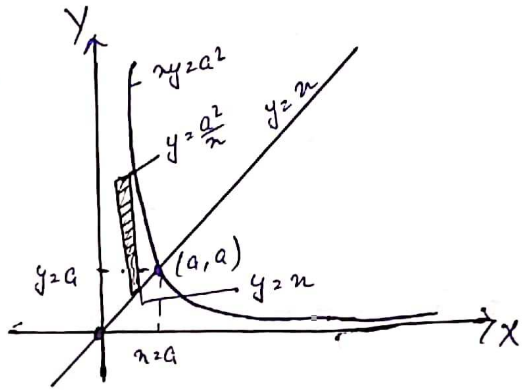
$\boxed{I = \frac{1}{2}}$ Ans

Q10 Change the order of integration :- $\int_0^a \int_x^{a^2/x} \phi(x,y)\, dx\, dy$.

Soln! Given Limit :- $0 \le x \le a$; $x \le y \le \frac{a^2}{x}$

$\Rightarrow y = x$ (Straight line)

$xy = a^2$ (Rectangular Hyperbola)

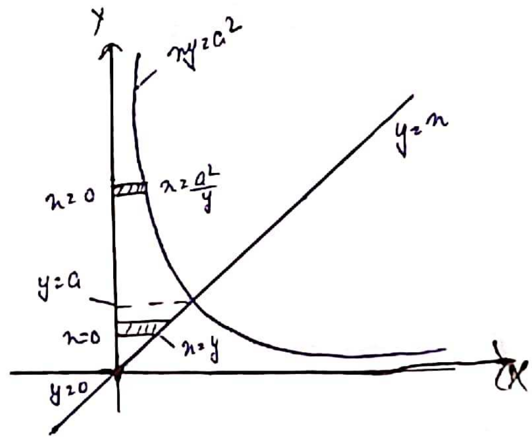


After change of order of integration,

$0 \le y \le a$ & $a \le y \le \infty$

$0 \le x \le y$ & $0 \le x \le \frac{a^2}{y}$

$\boxed{I = \int_0^a \int_0^y \phi(x,y)\, dx\, dy + \int_0^\infty \int_0^{a^2/y} \phi(x,y)\, dx\, dy.}$ Ans



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2019

Q5 Evaluate the triple integral $\int_0^1 \int_1^2 \int_2^3 dx\,dy\,dz$

Soln:- Let $I = \int_{z=0}^{1} \int_{y=1}^{2} \int_{x=2}^{3} dx\,dy\,dz$

$$I = \int_{z=0}^{1} \int_{y=1}^{2} (3-2)\,dy\,dz$$

$$I = \int_{z=0}^{1} \int_{y=1}^{2} 1 \cdot dy\,dz$$

$$I = \int_0^1 [y]_1^2\,dz$$

$$I = \int_0^1 [2-1]\,dz$$

$$I = \int_0^1 1 \cdot dz \Rightarrow [z]_0^1 \Rightarrow 1-0 \Rightarrow 1 \text{ Ans}$$

Q7 Find the area of the region bounded by the circle $x^2+y^2=a^2$, by double integration.

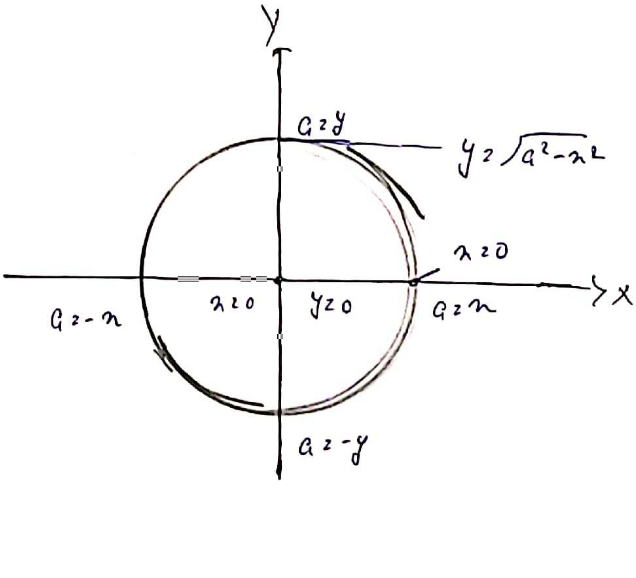
Soln:- Points:- $x^2+y^2=a^2$

$$x^2 = a^2-y^2$$

$$x = \pm\sqrt{a^2-y^2}$$

Limits:- $-\sqrt{a^2-y^2} \leq x \leq \sqrt{a^2-y^2}$

$$-a \leq y \leq a.$$



$$\Rightarrow \iint_R dx\,dy \Rightarrow \int_{y=-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 1 \cdot dx\,dy$$

$$\Rightarrow \int_{-a}^{a} \left[2\int_0^{\sqrt{a^2-y^2}} 1 \cdot dx\right] dy = 2\int_{-a}^{a} [x]_0^{\sqrt{a^2-y^2}}\,dy$$

$$\Rightarrow 2\int_{-a}^{a} \sqrt{a^2-y^2}\,dy \Rightarrow 2 \cdot 2\int_0^a \sqrt{a^2-y^2}\,dy$$

$$\Rightarrow 4\left[\frac{y\sqrt{a^2-y^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

$$\Rightarrow 4\left[0+\frac{a^2}{2}\sin^{-1}1\right]$$

$$\Rightarrow 4 \cdot \frac{1}{2}a^2 \cdot \frac{\pi}{2} \Rightarrow \boxed{\pi a^2} \text{ Ans}$$

Q13 (i) Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dx \, dy$. Also mention the region of integration involved in this double integral.

Soln:- Let $I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} x^2 y \, dy \, dx$

$$I = \int_0^a x^2 \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

Again diff. it w.r.t 'x'

$$\Rightarrow I = \frac{1}{2} \int_0^a x^2 (a^2 - x^2) \, dx$$

$$\Rightarrow I = \frac{1}{2} \int_0^a a^2 \cdot x^2 - x^4 \, dx$$

$$\Rightarrow I = \frac{1}{2} \left[a^2 \frac{x^3}{3} - \frac{x^5}{5} \right]_0^a$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{a^5}{3} - \frac{a^5}{5} \right]$$

$$\Rightarrow I = \frac{a^5}{2} \left[\frac{5-3}{15} \right]$$

$$\Rightarrow I = \frac{a^5}{\cancel{2}} \times \frac{\cancel{2}}{15} \Rightarrow \boxed{\frac{a^5}{15}} \text{ Ans}$$

Q13 (iii) Prove that the value of triple integration:-

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx \text{ is } \frac{1}{48}.$$

Soln: L.H.S Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy \, dx$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2} xy (1 - x^2 - y^2) \, dy \, dx$$

$$I = \int_0^1 \frac{1}{2} x \left[(1-x^2) \frac{y^2}{2} - \frac{y^4}{4} \right]_0^{\sqrt{1-x^2}} dx$$

$$I = \int_0^1 \frac{x}{2} \left[\frac{1}{2}(1-x^2)^2 - \frac{1}{4}(1-x^2)^2 \right] dx$$

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$$I = \int_0^1 \frac{x}{2} \cdot \frac{1}{4}(1-x^2)^2 dx$$

$$I = \frac{1}{8} \int_0^1 x(1-x^2)^2 \, dx$$

$$I = \frac{1}{8} \int_0^1 x(1+x^4-2x^2) \, dx$$

$$I = \frac{1}{8} \int_0^1 x + x^5 - 2x^3 \, dx$$

$$I = \frac{1}{8} \left[\frac{x^2}{2} + \frac{x^6}{6} - \frac{2x^4}{4} \right]_0^1$$

$$I = \frac{1}{8} \left[\frac{1}{2} + \frac{1}{6} - \frac{2}{4} \right]$$

$$I = \frac{1}{8} \left[\frac{1}{6} \right] \Rightarrow \boxed{I = \frac{1}{48}}$$ Ans

UNIT IV University Questions

27 March 2020 08:31 AM

2015

4. If $u = x \log y$, show that 3

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

7. Verify Euler's theorem when 7½

$f(x,y,z) = 3x^2yz + 5xy^2z + 4z^4$

11. (a) Determine the points where the function x^3+y^3-3axy has a maximum or minimum. 7½

(b) If $u = f(y-z, z-x, x-y)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$ 7½

2016

3. Find $\frac{\partial f}{\partial x}$, if $f = ye^{(x^2+y^2)}$.

7. If $u = \sin^{-1}\left\{\frac{x^2 + y^2}{x + y}\right\}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

11. (a) If $x = r\cos\theta$, $y = r\sin\theta$, show that :

(i) $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$

(ii) $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$

(b) Find the minima and maxima of $xy(a - x - y)$.

2017

4. If U = f(y/x), show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

8. Change the independent variable x to z in the equation

$(1 + x^2)^2 \frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} + y = x$ by the substitution $x = \tan z$

11. (i) If V=f (x-y, y-z, z-x), then prove that

$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0$

(ii) If $u = \log \dfrac{x^4 + y^4}{x + y}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

2018

4. If $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$ 3

8. Show that $\sin x(1+\cos x)$ is a maximum at $x = \dfrac{\pi}{3}$. 7½

13. (a) Transform the equation $x^4\left(\dfrac{d^2y}{dx^2}\right) + a^2 y = 0$

by the substitution $x = \dfrac{1}{z}$. 7½

(b) If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, show that : 7½

$$f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right).$$

2019

4. If $u = f\left(\frac{y}{x}\right)$ then prove that :

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

11. (i) Discuss the maxima or minima of the function :

$$u = xy + \left(\frac{a^3}{x}\right) + \left(\frac{a^3}{y}\right)$$

(ii) If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ then prove that :

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

2015

4. If $u = x \log y$, show that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ 3M

Soln. According to question,

L.H.S $\Rightarrow \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$

$= \dfrac{\partial}{\partial x}\left(\dfrac{x}{y}\right)$

$$\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{y}} \text{ — (A)}$$

R.H.S $\Rightarrow \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$= \dfrac{\partial}{\partial x}\left(\log y\right)$

$$\boxed{\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{1}{y}} \text{ — (B)}$$

from A and B

L.H.S = R.H.S Hence proved

7. Verify Euler's theorem when 7½ M

$f(x,y,z) = 3x^2yz + 5xy^2z + 4z^4$

Soln. Let $u = 3x^2yz + 5xy^2z + 4z^4$ — (A)

It is a homogeneous function with degree 4.

i.e. $n = 4$.

By Euler's theorem :

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n.u \quad \text{— (B)}$$

$$\frac{\partial u}{\partial x} = 6xyz + 5y^2z$$

$$\frac{\partial u}{\partial y} = 3x^2z + 10xyz$$

$$\frac{\partial u}{\partial z} = 3x^2y + 5xy^2 + 16z^3$$

According to equation (B).

L.H.S =

$$x(6xyz + 5y^2z) + y(3x^2z + 10xyz) + z(3x^2y + 5xy^2 + 16z^3)$$

$$= 6x^2yz + 5xy^2z + 3x^2yz + 10xy^2z + 3x^2yz + 5xy^2z + 16z^4$$

$$= 12x^2yz + 20xy^2z + 16z^4$$

$$= 4(3x^2yz + 5xy^2z + 4z^4)$$

$$= 4.u$$

Hence,

Euler's theorem verified.

3. 11.(a) Determine the points where the function $x^3 + y^3 - 3axy$ has a maximum or minimum.

Soln. Let $u = x^3 + y^3 - 3axy$

Step I → Differentiating u partially :

w.r.t $x \rightarrow p = \frac{\partial u}{\partial x} = 3x^2 - 3ay$

w.r.t $y \rightarrow q = \frac{\partial u}{\partial y} = 3y^2 - 3ax$

Step II →

putting $p = 0$	putting $q = 0$
i.e., $\frac{\partial u}{\partial x} = 0$	i.e., $\frac{\partial u}{\partial y} = 0$
$3x^2 - 3ay = 0$	$3y^2 - 3ax = 0$
$x^2 - ay = 0$	$y^2 - ax = 0$
$y = \frac{x^2}{a}$ — ①	$x = \frac{y^2}{a}$ — ②
from ②	from ③ putting $y = a$
$\frac{y^4}{a^2} - ay = 0$	$\Rightarrow x = \frac{a^2}{a}$
$y^4 - a^3 y = 0$	$\Rightarrow x = a$
$y(y^3 - a^3) = 0$	
$y = 0$ & $y^3 - a^3 = 0$	
$y = 0$ & $y = a$ — ③	

The points are (a, a) & $(0,0)$ where we will not consider $(0,0)$

Step III → $r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 6x$

$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -3a$

$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 6y$

Now, $rt - s^2 = 36xy - 9a^2$

Checking maximum & minimum on critical point:

at (a,a) i.e. $x = a$, $y = a$

$$rt - s^2 = 36 \times a \times a - 9a^2$$
$$= 36a^2 - 9a^2$$
$$= 27a^2 > 0$$

$\therefore$ It has minima & maxima at (a,a)

because $rt - s^2 > 0$

it gives maxima if $a < 0$ i.e. $r < 0$

it gives minima if $a > 0$ i.e. $r > 0$

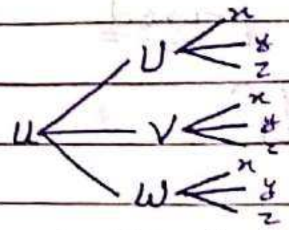
as $r = 6a$.

(b) If $u = f(y-z, z-x, x-y)$, prove that $7\frac{1}{2}$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Soln. Let $U = y-z$, $V = z-x$, $W = x-y$

Now, $u = f(U, V, W)$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial U}\frac{\partial U}{\partial x} + \frac{\partial u}{\partial V}\frac{\partial V}{\partial x} + \frac{\partial u}{\partial W}\frac{\partial W}{\partial x}$$

$$= \frac{\partial f}{\partial U}.(0) + \frac{\partial f}{\partial V}(-1) + \frac{\partial f}{\partial W}(1)$$

$$= \frac{\partial f}{\partial W} - \frac{\partial f}{\partial V}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial U}\frac{\partial U}{\partial y} + \frac{\partial u}{\partial V}\frac{\partial V}{\partial y} + \frac{\partial u}{\partial W}\frac{\partial W}{\partial y}$$

$$= \frac{\partial f}{\partial U}(+1) + \frac{\partial f}{\partial V}(0) + \frac{\partial f}{\partial W}.(-1)$$

$$= -\frac{\partial f}{\partial V} + \frac{\partial f}{\partial U} = \frac{\partial f}{\partial U} - \frac{\partial f}{\partial V}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial U}\frac{\partial U}{\partial z} + \frac{\partial u}{\partial V}\frac{\partial V}{\partial z} + \frac{\partial u}{\partial W}\frac{\partial W}{\partial z}$$

$$= \frac{\partial f}{\partial U}(-1) + \frac{\partial f}{\partial V}(1) + \frac{\partial f}{\partial W}(0)$$

$$= \frac{\partial f}{\partial W} - \frac{\partial f}{\partial U}$$

According to question L.H.S $= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$$= \frac{\partial f}{\partial w} - \frac{\partial f}{\partial v} + \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w} + \frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}$$

$$= 0 = R.H.S$$

Hence proved i.e. L.H.S = R.H.S

2016

3. Find $\frac{\partial f}{\partial x}$, if $f = ye^{(x^2+y^2)}$.

Soln. $f = ye^{(x^2+y^2)}$

Differentiating partially both side w.r.t x.

$$\frac{\partial f}{\partial x} = y\, e^{(x^2+y^2)} \cdot 2x$$

$$= 2xy\, e^{(x^2+y^2)}$$

7. If $u = \sin^{-1}\left\{\frac{x^2+y^2}{x+y}\right\}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

Soln. Let $g(u) = \sin u = \left\{\frac{x^2+y^2}{x+y}\right\}$

Checking for homogeneous function i.e.

$$f(x,y) = \frac{x^2+y^2}{x+y} \Rightarrow f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{\lambda x + \lambda y}$$

$$= \frac{\lambda^2 (x^2+y^2)}{\lambda (x+y)}$$

$$= \lambda f(x,y)$$

$g(u)$ is a homogeneous function with degree 1.

i.e. $n = 1$.

By deduction of Euler's Theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot \frac{g(u)}{g'(u)}$$

$\Rightarrow x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1 \cdot \dfrac{\sin u}{\cos u}$

$\Rightarrow \boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u}$

11 (a) If $x = r\cos\theta$, $y = r\sin\theta$, show that :

(i) $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$

soln $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$ — ①

Differentiating partially w.r.t x & y.

$2x + 0 = 2r\dfrac{\partial r}{\partial x}$ & $0 + 2y = 2r\dfrac{\partial r}{\partial y}$

$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ & $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

According to question, $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2$

$= \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2}$

$= \dfrac{x^2 + y^2}{r^2}$

from ① $= \dfrac{r^2}{r^2} = 1 = R.H.S$

Hence proved, $\boxed{L.H.S = R.H.S}$

8. (ii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

Soln. $x = r\cos\theta$

We have $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

$\frac{y}{x} = \tan\theta$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \theta$

$\therefore \frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2} \cdot \left(\frac{-y}{x^2}\right)$

$= \frac{-y}{x^2 + y^2}$

Again we have $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2}$

$\therefore \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{2xy}{(x^2+y^2)^2}$ — ①

also $\dfrac{\partial \theta}{\partial y} = \dfrac{1}{1+y^2/x^2} \cdot \dfrac{1}{x} = \dfrac{x}{x^2+y^2}$

$\therefore \dfrac{\partial^2 \theta}{\partial y^2} = \dfrac{-2xy}{x^2+y^2}$ — ②

According to question.

$L.H.S = \dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2}$

$= \dfrac{2xy}{(x^2+y^2)^2} + \left(\dfrac{-2xy}{x^2+y^2}\right)$ {∵ from ① & ②}

$= 0 = R.H.S$

Hence proved that $L.H.S = R.H.S$

9. (b) Find the minima and maxima of $xy(a-x-y)$

Soln. let $u = xy(a-x-y)$

$= axy - x^2y - xy^2$

Step I → Differentiating partially :-

w.r.t $x \rightarrow$ $p = \dfrac{\partial u}{\partial x} = ay - 2xy - y^2$

w.r.t $y \rightarrow$ $q = \dfrac{\partial u}{\partial y} = ax - x^2 - 2xy$

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Step 2 →

putting $p = 0$

i.e., $\frac{\partial u}{\partial x} = 0$

$ay - 2xy - y^2 = 0$

$y(a - 2x - y) = 0$

$y = 0$ & $a - 2x - y = 0$

putting $q = 0$

i.e., $\frac{\partial u}{\partial y} = 0$

$ax - x^2 - 2xy = 0$

$x(a - x - 2y) = 0$

$x = 0$ & $a - x - 2y = 0$

For critical points we have 3 combinations of above equations :-

(i) $y = 0$ & $a - x - 2y = 0$

$a - x - 2 \times 0 = 0$

$a - x = 0$

$x = a$

point is $(a, 0)$

(ii) $a - 2x - y = 0$ & $x = 0$

$a - 2 \times 0 - y = 0$

$a - y = 0$

$y = a$

point is $(0, a)$

(iii) $a - 2x - y = 0$ & $a - x - 2y = 0$

$a - 2x - y = 0$

$-y = -a + 2x$

$y = a - 2x$ — ①

from ②

$y = a - 2(a - 2y)$

$y = a - 2a + 4y$

$3y = a$

$y = \frac{a}{3}$

$a - x - 2y = 0$

$-x = -a + 2y$

$x = a - 2y$ — ②

from ①

$x = a - 2(a - 2x)$

$x = a - 2a + 4x$

$3x = a$

$x = \frac{a}{3}$

The critical points are $(0,a), (a,0), \left(\frac{a}{3}, \frac{a}{3}\right)$.

Step-3 →

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -2y$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = a - 2x - 2y$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -2x$$

Now, $rt - s^2 = 4xy - (a - 2x - 2y)^2$

Checking maxima & minima on critical point :-

at $(0,a)$ i.e. $x = 0$ & $y = a$

$$rt - s^2 = 4 \times 0 \times a - (a - 2\times 0 - 2\times a)^2$$
$$= -(a - 2a)^2 = -a^2$$

∴ no maxima & minima at $(0,a)$ because $rt - s^2 < 0$

at $(a,0)$ i.e. $x = a$ & $y = 0$

$$rt - s^2 = 4 \times a \times 0 - (a - 2\times a - 2\times 0)^2$$
$$= -(a - 2a)^2 = -a^2$$

∴ neither maxima nor minima at $(a,0)$ because $rt - s^2 < 0$

at $(a/3, a/3)$ i.e. $x = a/3$ & y $a/3$

$$rt - s^2 = 4 \times \frac{a}{3} \times \frac{a}{3} - \left(a - \frac{2a}{3} - \frac{2a}{3}\right)^2$$

$$= \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3}$$

$\therefore$ It has minima and maxima at $(a/3, a/3)$
because $rt - s^2 > 0$
it gives maxima if $a > 0$ i.e. $r < 0$
it gives minima if $a < 0$ i.e. $r > 0$
as $r = -2a$

2017

4. If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Soln. $u = f\left(\frac{y}{x}\right)$

Differentiate u partially :-

w.r.t. $x \rightarrow \frac{\partial u}{\partial x} = f'\cdot\left(\frac{-y}{+x^2}\right)$

w.r.t. $y \rightarrow \frac{\partial u}{\partial y} = f'\left(\frac{1}{x}\right)$

According to question,

$$L.H.S = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \cancel{x}\left(\frac{-yf'}{x^{\cancel{2}}}\right) + \frac{yf'}{x}$$

$$= \frac{-yf'}{x} + \frac{yf'}{x}$$

$$= 0 = R.H.S.$$

Hence proved i.e. $L.H.S = R.H.S.$

Q. Change the independent variable x to z in the equation

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + y = x \quad -①$$ by the substitution $x = \tan z$

Soln. $x = \tan z$

differentiating w.r.t. x

$$1 = \sec^2 z \frac{dz}{dx} \Rightarrow \frac{dz}{dx} = \cos^2 z \quad —②$$

Again diff. w.r.t. 'x'

$$\frac{d^2z}{dx^2} = -2\cos z \sin z \frac{dz}{dx}$$

$$\frac{d^2z}{dx^2} = -2\cos z \sin z \cos^2 z$$

$$\frac{d^2z}{dx^2} = -2\sin z \cos^3 z \quad —③$$

from ①

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + y = x$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \cos^2 z \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d^2z}{dx^2}\frac{dy}{dz}$$

$$= \frac{d^2y}{dz^2}\left(\cos^2 z\right)^2 + (-2\sin z \cos^3 z)\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \cos^4 z \frac{d^2y}{dz^2} - 2\sin z \cos^3 z \frac{dy}{dz}$$

Again from ①

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + y = x$$

$$(1+\tan^2 z)^2\left[\cos^4 z\,\frac{d^2y}{dz^2} - 2\sin z\cos^3 z\,\frac{dy}{dz}\right] + 2\tan z\,(1+\tan^2 z)\cos^2 z\frac{dy}{dz} + y = \tan z$$

$\because\ 1+\tan^2 z = \sec^2 z$

$$(\sec^2 z)^2\left[\cos^4 z\,\frac{d^2y}{dz^2} - 2\sin z\cos^3 z\,\frac{dy}{dz}\right] + 2\tan z\sec^2 z\cos^2 z\frac{dy}{dz} + y = \tan z$$

$$\frac{d^2y}{dz^2} - 2\tan z\frac{dy}{dz} + 2\tan z\frac{dy}{dz} + y = \tan z$$

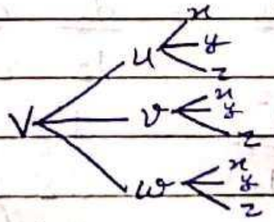
$$\boxed{\frac{d^2y}{dz^2} + y = \tan z}$$

12 11(P) If $V = f(x-y,\ y-z,\ z-x)$, then prove that

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} = 0$$

Soln. Let $u = x-y$, $v = y-z$, $w = z-x$

Now, $V = f(u, v, w)$



$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial V}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1)$$

$$= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial V}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0)$$

$$= \frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial V}{\partial w}\frac{\partial w}{\partial z}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1)$$

$$= \frac{\partial f}{\partial w} - \frac{\partial f}{\partial v}$$

According to question

$$L.H.S = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}$$

$$= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w} + \frac{\partial f}{\partial v} - \frac{\partial f}{\partial u} + \frac{\partial f}{\partial w} - \frac{\partial f}{\partial v}$$

$$= 0 = R.H.S$$

Hence proved i.e. L.H.S = R.H.S .

13 (ii) If $u = \log \frac{x^4+y^4}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

Soln. let $g(u) = e^u = \frac{x^4+y^4}{x+y}$

Checking that function is homogeneous or not :-

$$f(x,y) = \frac{x^4+y^4}{x+y} \Rightarrow f(\lambda x, \lambda y) = \frac{\lambda^4 x^4 + \lambda^4 y^4}{\lambda x + \lambda x}$$

$$= \lambda^3 \left[\frac{x^4+y^4}{x+y}\right]$$