Vector Differentiation

6.1 Vector Point Functions

The values of vector valued functions are vectors of the form

$$\mathbf{F(p)} = F_1(p)\,\hat{\mathbf{i}} + \mathbf{F_2}(p)\,\hat{\mathbf{j}} + \mathbf{F_3}(p)\,\hat{\mathbf{k}} \qquad \text{...(1)}$$

The values depend on the point P is space. A vector valued function defined a **vector** field in the region or on that surface or curve. This function may also depend on time t or any other parameter.

1. Equation in cartesian co-ordinates (x,y,z) can be written as

$$F(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{i}} + F_2(x,y,z)\,\hat{\mathbf{j}} + F_3(x,y,z)\,\hat{\mathbf{k}}$$

Illustration:

Let $F(t) = a\cos t\,\hat{\mathbf{i}} + b\sin t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ is a vector function of the scalar variable t, where $F_1(t) = a\cos t$, $F_2(t) = b\sin t$, $F_3(t) = t^2$. This **F** is a vector function and $\mathbf{F}(t)$ is a vector quantity. $a\cos t$, $b\sin t$, t are called components of the vector $\mathbf{F}(t)$ along the co-ordinate axes and $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be unit vectors along x, y and z axis.

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6.3 Rules for Differentiation

Let $\mathbf{f}(t)$, $\mathbf{g}(t)$ and $\mathbf{h}(t)$ be three vector functions of scalar t. Then,

1. $\dfrac{d}{dt}[\mathbf{f}(t) \pm \mathbf{g}(t)] = \dfrac{d\mathbf{f}(t)}{dt} \pm \dfrac{d\mathbf{g}(t)}{dt}$

2. $\dfrac{d}{dt}[c\,\mathbf{f}(t)] = c\dfrac{d\mathbf{f}(t)}{dt}$, where c is scalar constant.

3. $\dfrac{d}{dt}[\mathbf{f}(t).\mathbf{g}(t)] = \dfrac{d}{dt}[\mathbf{f}(t)].\mathbf{g}(t) + \mathbf{f}(t).\dfrac{d\mathbf{g}(t)}{dt}$

4. $\dfrac{d}{dt}[\mathbf{f}(t)\times\mathbf{g}(t)] = \dfrac{d}{dt}[\mathbf{f}(t)]\times\mathbf{g}(t) + \mathbf{f}(t)\times\dfrac{d}{dt}[\mathbf{g}(t)]$

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...ple 1: Show that the derivative of a vector of constant length is either zero vector or is ...dicular to the vector.

...tion: Let f(t) be the vector of constant length

$$|\mathbf{f}(t)| = c$$

$$|f(t)|^2 = \mathbf{f}(t).\mathbf{f}(t) = c^2$$

...hen,

$$\frac{d}{dt}[\mathbf{f}(t).\mathbf{f}(t)] = \frac{d}{dt}[\mathbf{f}(t)].\mathbf{f}(t) + \mathbf{f}(t).\frac{d}{dt}[f(t)] = 2\,\mathbf{f}(t).\frac{d\,f(t)}{dt} = 0$$

$$\mathbf{f}(t).\frac{df(t)}{dt} = 0$$

either $\mathbf{f}(t)=0$ or $\frac{d}{dt}(\mathbf{f}(t))=0$ or $\frac{d}{dt}[\mathbf{f}(t)]$ is perpendicular to $\mathbf{f}(t)$.

6.4 Velocity and Acceleration

Let r be a vector function of the scalar variable t.

Then,

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \qquad ...(1)$$

be position vector of a point, where x,y,z be components along x, y and z axis.

Differentiate w.r.t (t) to (1), we get velocity (v)

$$v = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

$\therefore$ $\frac{d\,\hat{\mathbf{i}}}{dt} = \frac{d\,\hat{\mathbf{j}}}{dt} = \frac{d\,\hat{\mathbf{k}}}{dt} = 0$ as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be constant unit vector.

Where $\frac{dx}{dt}$ = component of velocity along x-axis

$\frac{dy}{dt}$ = component of velocity along y-axis

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Theorem 1: The necessary and sufficient condition for the vector $\mathbf{F}(t)$ has a constant direction is that

$$\mathbf{F} \times \frac{d\mathbf{F}}{dt} = 0.$$

[B.C.A. (Bhopal), 2012, 09, 08; B.C.A. (Agra) 2008, 06, 04
B.C.A. (Kanpur) 2006; B.C.A. (Meerut) 2001]

Proof: Let $|\mathbf{F}(t)| = f(t)$. Let $\mathbf{G}(t)$ be unit vector in the direction of $\mathbf{F}(t)$. Then,

$$\frac{\mathbf{F}(t)}{|\mathbf{F}(t)|} = \mathbf{G}(t)$$

$$\Rightarrow \quad \mathbf{F}(t) = f(t)\mathbf{G}(t) \quad \text{...(1)}$$

Differentiate w.r.t. (t)

$$\frac{d\mathbf{F}(t)}{dt} = f(t)\,\frac{d\mathbf{G}(t)}{dt} + \frac{d\,f(t)}{dt}\mathbf{G}(t)$$

Since $\mathbf{G}(t)$ has constant direction, then $\frac{d\mathbf{G}(t)}{dt} = 0$

$$\Rightarrow \quad \frac{d\,\mathbf{F}(t)}{dt} = 0 + \frac{df(t)}{dt}\mathbf{G}(t)$$

Taking cross product with $\mathbf{F}(t)$ on both sides, we find

$$\mathbf{F} \times \frac{d\,\mathbf{F}(t)}{dt} = f(t)\mathbf{G}(t) \times \frac{df(t)}{dt}\mathbf{G}(t)$$

$$\Rightarrow \quad \mathbf{F} \times \frac{d\,\mathbf{F}(t)}{dt} = 0 \qquad \text{But } [\mathbf{G}(t) \times \mathbf{G}(t)] = 0$$

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Example 2: *A particle moves along the curve*

$$\mathbf{r} = (t^3 - 4t)\,\hat{\mathrm{i}} + (t^2 + 4t)\,\hat{\mathrm{j}} + (8t^2 - 3t^3)\,\hat{\mathrm{k}}$$

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where t is the time. Find the magnitude of the tangential components of its acceleration at $t = 2$.

[B.C.A. (Meerut) ...

Solution: We have

$$\mathbf{r} = (t^3 - 4t)\,\hat{\mathbf{i}} + (t^2 + 4t)\,\hat{\mathbf{j}} + (8t^2 - 3t^3)\,\hat{\mathbf{k}}$$

$\therefore$ $$\text{Velocity} = \frac{d\mathbf{r}}{dt} = (3t^2 - 4)\,\hat{\mathbf{i}} + (2t + 4)\,\hat{\mathbf{j}} + (16t - 9t^2)\,\hat{\mathbf{k}}$$

At $t = 2$, $$\text{Velocity} = 8\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$$

$$\text{Acceleration} = \mathbf{a} = \frac{d^2 r}{dt^2} = 6t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (16 - 18t)\,\hat{\mathbf{k}}$$

At $t = 2$, $$\text{Acceleration} = \mathbf{a} = 12\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 20\,\hat{\mathbf{k}}$$

We know the velocity of particle on curve is the tangent to the curve. So, the tangent vector is velocity. Then, unit tangent vector

$$\hat{\mathbf{t}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{8\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{\sqrt{64 + 64 + 16}}$$

$$= \frac{8\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{12}$$

$$= \frac{2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}}{3}$$

Tangential component of acceleration $= \mathbf{a} \cdot \hat{\mathbf{t}}$

$$= (12\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 20\,\hat{\mathbf{k}}) \cdot \left(\frac{2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}}{3}\right)$$

$$= \frac{24 + 4 + 20}{3} = \frac{48}{3} = 16$$

Example 3: *A particle moves along the curve* $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ *where t is time. Find the components of its velocity and acceleration at time* $t = 1$ *in the direction*

$$\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}.$$

[B.C.A. (Rohilkhand) 2009; B.C.A. (Avadh) 2008]

Solution: Let $$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

$\Rightarrow$ $$\mathbf{r} = 2t^2\,\hat{\mathbf{i}} + (t^2 - 4t)\,\hat{\mathbf{j}} + (3t - 5)\,\hat{\mathbf{k}}$$

$\therefore$ $$\text{Velocity} = \frac{d\mathbf{r}}{dt} = 4t\,\hat{\mathbf{i}} + (2t - 4)\,\hat{\mathbf{j}} + (3)\,\hat{\mathbf{k}}$$

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Velocity $= 4\hat{i} - 2\hat{j} + 3\hat{k}$

at $t = 1$,

$$a = \hat{i} - 3\hat{j} + 2\hat{k}$$

Unit vector

$$\hat{a} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1+9+4}} = \frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}}$$

The component of the velocity in the given direction

$$= \frac{d\mathbf{r}}{dt}.\hat{\mathbf{a}}$$

$$= (4\hat{i} - 2\hat{j} + 3\hat{k}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}}$$

$$= \frac{8\sqrt{14}}{7}$$

$$\text{Acceleration} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = 4\hat{i} + 2\hat{j}$$

Then, the component of the acceleration in the given direction

$$a = \frac{d^2\mathbf{r}}{dt^2}.\hat{\mathbf{a}}$$

$$= (4\hat{i} + 2\hat{j}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}} = \frac{-\sqrt{14}}{7}.$$

Example 4: ***Show that if*** $\mathbf{r} = \mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t$. ***Where a, b, ω are constants, then***

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r} \text{ and } \mathbf{r} \times \frac{d\mathbf{r}}{dt} = -\omega\,\mathbf{a} \times \mathbf{b}.$$

[B.C.A. (Purvanchal) 2009]

Solution: We have

$$\mathbf{r} = \mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t \qquad ...(1)$$

$$\Rightarrow \quad \frac{d\mathbf{r}}{dt} = \mathbf{a}\omega\cos\omega t - \mathbf{b}\omega\sin\omega t$$

$$\Rightarrow \quad \frac{d^2\mathbf{r}}{dt^2} = -\mathbf{a}\omega^2\sin\omega t - \mathbf{b}\omega^2\cos\omega t$$

$$\Rightarrow \quad \frac{d^2\mathbf{r}}{dt^2} = -\omega^2(\mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t)$$

[by (1)]

$$\Rightarrow \quad \frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}$$

Also

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = (\mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t) \times (\mathbf{a}\omega\cos\omega t - \mathbf{b}\omega\sin\omega t)$$

$$= \omega(-(\mathbf{a}\times\mathbf{b})\sin^2\omega t + (b\times a)\cos^2\omega t) \qquad [\because \mathbf{a}\times\mathbf{a} = \mathbf{b}\times\mathbf{b} = 0]$$

$$= \omega(-(a\times b)\sin^2\omega t - (\mathbf{a}\times b)\cos^2\omega t)$$

$$= -\omega(\mathbf{a}\times\mathbf{b})[\sin^2\omega t + \cos^2\omega t]$$

$$= -\omega(\mathbf{a}\times\mathbf{b}).$$

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Example 8: If $\hat{\mathbf{r}}$ be the unit vector in the direction r, show that

$$\hat{\mathbf{r}} \times d\,\hat{\mathbf{r}} = \frac{\mathbf{r} \times d\mathbf{r}}{r^2}.$$

[B.C.A. (Meerut) 2007]

Solution:

$$d(\hat{\mathbf{r}}) = d\left(\frac{\mathbf{r}}{r}\right) = \frac{d\mathbf{r}}{r} - \mathbf{r}\frac{dr}{r^2}$$

$\therefore$

$$\hat{\mathbf{r}} \times d\hat{\mathbf{r}} = \hat{\mathbf{r}} \times \left(\frac{dr}{r} - \mathbf{r}\frac{dr}{r^2}\right)$$

$$= \frac{\mathbf{r}}{r}\left(\frac{d\mathbf{r}}{r} - \mathbf{r}\frac{dr}{r^2}\right) \qquad \left[\because \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}\right]$$

$$= \frac{\mathbf{r} \times d\mathbf{r}}{r^2} - (\mathbf{r} \times \mathbf{r})\frac{dr}{r^3}$$

$$= \frac{\mathbf{r} \times d\mathbf{r}}{r^2} - 0 \qquad [\because \mathbf{r} \times \mathbf{r} = 0]$$

$$= \frac{\mathbf{r} \times d\mathbf{r}}{r^2}$$

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Gradient, Divergence & Curl

7.1 Scalar and Vector Point Function

If a quantity assume one or more than one definite value at each point of a region, then that quantity is called **point function** in the given region.

7.2 Single Valued Function

A point function is said to be **single valued** or **uniform function** if it has only one definite value at each point of the region otherwise it is called **multiple valued function** or **multivalued function** in the given region.

7.2.1 Scalar Point Function

If corresponding to each point P of a region R, there is associated a scalar $\phi(P)$ or $\phi(x, y, z)$. Then the function $\phi(P)$ is called **scalar point function** or **scalar function of position** design in the region R.

Illustration:

1. The mass $m(P)$ at the point P of a body occupying a certain region is a scalar point function.

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7.3 Scalar Field

The set of points of the region R together with corresponding scalar function values is said to be a **scalar field** over R.

Illustration:

1. $\phi(x, y, z) = x^2 y - z^3$
2. Gravitational potential of system of masses
3. The temperature distribution in a medium

7.4 Vector Point Function

If corresponding to each point P of a region R, there is associated a vector $\mathbf{f}(P)$ than f is called vector point function or vector function of position defined in the region R.

Illustration:

1. The velocity $v(P)$ of a particle in a moving fluid at any time t occupying the position P in a certain region is **vector point functions.**

2. The gravitational force $G(P)$ or $F(x, y, z)$ exerted by a given point mass m at the origin on a unit point mass located at a point $P(x, y, z)$ other than the origin is

$$G(P) = \frac{Gm}{x^2 + y^2 + z^2} u(x, y, z)$$

where G = the universal gravitational constant.

$u(x, y, z)$ = unit vector emanating from P and directed toward the origin.

Here, $G(P)$ is a vector point function.

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1.9 Tangent Plane and Normal Plane to a Surface Level

1. The equation of tangent plane at $P(a,b,c)$ on surface $f(x,y,z)=C$ is

$$\nabla f(P)[(x-a)\,\hat{\mathbf{i}}+(y-b)\,\hat{\mathbf{j}}+(z-c)\,\hat{\mathbf{k}}]=0$$

or

$$(x-a)\frac{\partial f}{\partial x}+(y-b)\frac{\partial f}{\partial y}+(z-c)\frac{\partial f}{\partial z}=0$$

2. The equation of normal plane to the surface $f(x,y,z)=C$ at $P(a,b,c)$ is

$$\frac{(x-a)}{\dfrac{\partial f}{\partial x}}=\frac{(y-b)}{\dfrac{\partial f}{\partial y}}=\frac{(z-c)}{\dfrac{\partial f}{\partial z}}$$

3. Vector normal to surface $f(x,y,z)=C$ is ∇f.

4. The unit vector normal to surface $f(x,y,z)=C$ is

$$\hat{n}=\frac{\nabla f}{|\nabla f|}=\frac{\nabla f}{|grad\ f|}$$

5. Angle between two surface $\phi_1(x,y,z)=C_1$ and $\phi_2(x,y,z)=C_2$ is given by

$$\cos\phi=\frac{(\nabla\phi_1).(\nabla\phi_2)}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

where ϕ is the angle between $\phi_1\,(x,y,z)=C_1$ and $\phi_2\,(x,y,z)=C_2$ surfaces

$$\nabla\phi_1=\text{ grad }\phi_1=\frac{\partial\phi_1}{\partial x}\hat{\mathbf{i}}+\frac{\partial\phi_1}{\partial y}\hat{\mathbf{j}}+\frac{\partial\phi_1}{\partial z}\hat{\mathbf{k}}$$

and

$$\nabla\phi_2=\text{grad }\phi_2=\frac{\partial\phi_2}{\partial x}\hat{\mathbf{i}}+\frac{\partial\phi_2}{\partial y}\hat{\mathbf{j}}+\frac{\partial\phi_2}{\partial z}\hat{\mathbf{k}}$$

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7.10 Some Results Connected to Gradient

Theorem 1: Prove the following relations:

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(fg) = f\,\nabla g + g\,\nabla f$ [B.C.A. (Meerut) 2004]

(iii) $\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2},\ g \neq 0$

where f, g be two differentiable scalar fields.

Proof: (i) We have $\nabla(f \pm g) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(f \pm g)$

$$= \hat{i}\frac{\partial}{\partial x}(f \pm g) + \hat{j}\frac{\partial}{\partial y}(f \pm g) + \hat{k}\frac{\partial}{\partial z}(f \pm g)$$

$$= \hat{i}\left[\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}\right] + \hat{j}\left[\frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y}\right] + \hat{k}\left[\frac{\partial f}{\partial z} \pm \frac{\partial g}{\partial z}\right]$$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \pm \left(\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right)$$

$$= \nabla f \pm \nabla g$$

$\therefore$ $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) Similar proof as proof (1).

(iii) We have $\nabla\left(\frac{f}{g}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{f}{g}\right)$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$

$$= \hat{i}\left\{\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right\} + \hat{j}\left\{\frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\right\} + \hat{k}\left\{\frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right\}$$

$$= \frac{1}{g^2}\left\{g\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) - f\left(\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right)\right\}$$

$$= \frac{g\,\nabla f - f\,\nabla g}{g^2}$$

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Example 1: If $\phi = 3x^2 y - y^3 z^2$, then find grad ϕ at the point $(1, -2, -1)$.

[B.C.A. (Meerut) 2007,06]

Solution: Here,

$$\phi = 3x^2 y - y^3 z^2$$

$$\frac{\partial \phi}{\partial x} = 6xy, \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2 z^2, \frac{\partial \phi}{\partial z} = -2y^3 z$$

$$\text{grad } \phi = \nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2 z^2)\,\hat{j} + (-2y^3 z)\,\hat{k} \text{ at } (1, -2, -1)$$

$$= -12\,\hat{i} - 9\,\hat{j} - 16\,\hat{k}$$

Example 2: If $r = |\mathbf{r}|$, where $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, prove that

(i) $\nabla f(r) = f'(r)\nabla \mathbf{r}$ [B.C.A. (Meerut) 2012,05,01]

(ii) $\nabla r = \frac{\mathbf{r}}{r}$ [B.C.A. (Agra) 2008,05]

(iii) $\nabla f(r) \times \mathbf{r} = 0$ [B.C.A. (Meerut) 2003]

(iv) $\nabla\left(\frac{1}{r}\right) = \frac{-\mathbf{r}}{r^3}$ [B.C.A. (Meerut) 2004]

(v) $\nabla \log|r| = \frac{\mathbf{r}}{r^2}$ [B.C.A. (Meerut) 2011]

(vi) $\nabla r^n = n r^{n-2}\,\mathbf{r}$ [B.C.A. (Lucknow) 2011,06]

[B.C.A. (Avadh) 2009]

(vii) $\nabla |r^2| = 2\mathbf{r}$

(viii) $\nabla e^{(x^2 + y^2 + z^2)} = 2e^{r^2}\,\mathbf{r}$.

Solution: If

$$\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Then

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \quad \text{or} \quad r^2 = x^2 + y^2 + z^2 \quad \ldots(1)$$

Differentiate partially w.r.t. x, y and z, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

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7.11 Directional Derivative

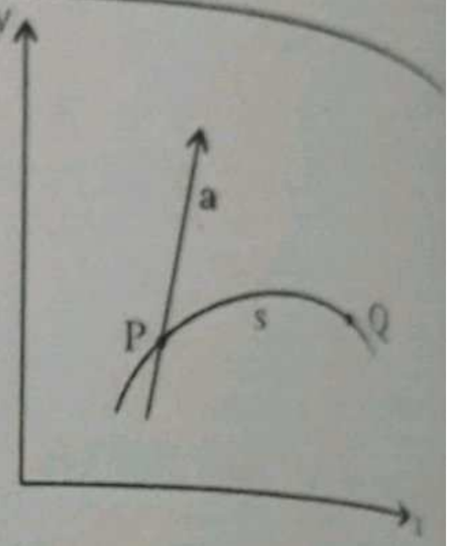

Fig. 7.4: Directional derivative

The rate of change of a scalar function ϕ at any point P in any fixed direction **a** is called the **directional derivative** of ϕ at P in the direction **a** and is denoted by $\frac{d\phi}{ds}$.

From advance calculus, we have

$$\nabla\phi = \frac{\partial \phi}{\partial x}\Delta x + \frac{\partial \phi}{\partial y}\Delta y + \frac{\partial \phi}{\partial z}\Delta z + \text{terms of higher}$$

powers of $\Delta x, \Delta y, \Delta z$

Then, $$\lim_{\Delta s \to 0}\frac{\Delta\phi}{\Delta s} = \lim_{\Delta s \to 0}\left(\frac{\partial \phi}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial \phi}{\partial y}\frac{\Delta y}{\Delta s} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial s}\right)$$

or $$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\hat{\mathbf{i}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial \phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial \phi}{\partial z}\right) \cdot \left(\hat{\mathbf{i}}\frac{dx}{ds} + \hat{\mathbf{j}}\frac{dy}{ds} + \hat{\mathbf{k}}\frac{dz}{ds}\right)$$

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7.12 Gradient of Constant Vector

If ϕ is constant, then $\quad \dfrac{\partial \phi}{\partial x}=0, \dfrac{\partial \phi}{\partial y}=0, \dfrac{\partial \phi}{\partial z}=0$

$$\text{grad } \phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}$$

$$= 0 . \hat{\mathbf{i}} + 0 . \hat{\mathbf{j}} + 0 . \hat{\mathbf{k}}$$

$$= 0$$

Thus, $\nabla\phi = 0 \Leftrightarrow$ function is constant.

Example 7: ***Find the unit normal $\hat{n}$ of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point $P(1,0,2)$***

Solution: Let $\quad f = z^2 - 4x^2 - 4y^2 = 0 \qquad ...(1)$

From (1) $\quad \dfrac{\partial f}{\partial x} = -8x, \dfrac{\partial f}{\partial y} = -8y, \dfrac{\partial f}{\partial z} = 2z$

$\therefore \quad \nabla f = \dfrac{\partial f}{\partial x}\hat{\mathbf{i}} + \dfrac{\partial f}{\partial y}\hat{\mathbf{j}} + \dfrac{\partial f}{\partial z}\hat{\mathbf{k}}$

$$= -8x\,\hat{\mathbf{i}} - 8y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}} \text{ at } (1,0,2)$$

$$\nabla f = -8\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}$$

$\Rightarrow \quad |\nabla f| = \sqrt{(-8)^2 + (0)^2 + (4)^2} = \sqrt{64 + 0 + 16} = \sqrt{80}$

The vector normal to surface $= \nabla f$

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Example 8: Find the equations of the tangent plane and normal plane to the surface $2xz^2-3xy-4x=7$ at the point $(1,-1,2)$. [B.C.A. (Bundelkhand) 2011,07]

Solution: Let

$$f=2xz^2-3xy-4x-7=0 \quad \text{...(1)}$$

be given surface

$$\frac{\partial f}{\partial x}=2z^2-3y-4, \frac{\partial f}{\partial y}=-3x, \frac{\partial f}{\partial z}=4xz$$

Find

At $(1,-1,2)$

$$\frac{\partial f}{\partial x}=7, \frac{\partial f}{\partial y}=-3, \frac{\partial f}{\partial z}=8$$

The equation of tangent plane to the surface $f(1)$ at which (x_1, y_1, z_1) is

$$(x-x_1)\frac{\partial f}{\partial x}+(y-y_1)\frac{\partial f}{\partial y}+(z-z_1)\frac{\partial f}{\partial z}=0$$

Then at $(1,-1,2)$

$$(x-1)(7)+(y+1)(-3)+(z-2)(8)=0$$

$$7x-3y+8z-7-3-16=0$$

$$7x-3y+8z=26$$

or

The equation of normal plane at (x_1, y_1, z_1) is

$$\frac{x-x_1}{\frac{\partial f}{\partial x}}=\frac{y-y_1}{\frac{\partial f}{\partial y}}=\frac{z-z_1}{\frac{\partial f}{\partial z}}$$

$$\therefore \quad \frac{x-1}{7}=\frac{y+1}{-3}=\frac{z-2}{8}$$

Example 9: Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Solution: Let

$$f_1=x^2+y^2+z^2-9=0 \quad \text{...(1)}$$

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7.13.2 Solenoidal Vector

A vector field defined by the vector function $\mathbf{F}(x, y, z)$ is called solenoidal, if div $(\mathbf{F}) = 0$

7.13.3 Divergence of a Constant Vector

Theorem 3: If A is constant vector, then div (A) = 0.

Proof: Since, **A** is constant vector

$$\Rightarrow \quad \frac{\partial A}{\partial x} = 0, \frac{\partial A}{\partial y} = 0, \frac{\partial A}{\partial z} = 0$$

Then,

$$\text{div}(\mathbf{A}) = \nabla . \mathbf{A}$$

$$= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).(A)$$

$$= \hat{\mathbf{i}}\frac{\partial A}{\partial x} + \hat{\mathbf{j}}\frac{\partial A}{\partial y} + \hat{\mathbf{k}}\frac{\partial A}{\partial z}$$

$$= \hat{\mathbf{i}}0 + \hat{\mathbf{j}}0 + \hat{\mathbf{k}}0$$

$$= 0$$

7.14 Curl of a Vector Point Function

Let $\mathbf{V}(x, y, z)$ be a differentiable vector function, where x, y and z cartesian coordinates. **The curl (or rotation)** of **V** is denoted by curl (**V**) and defined as

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V}$$

$$= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \mathbf{V}$$

$$= \hat{\mathbf{i}} \times \frac{\partial \mathbf{V}}{\partial x} + \hat{\mathbf{j}} \times \frac{\partial \mathbf{V}}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial \mathbf{V}}{\partial z}$$

Clearly, **the curl of a vector function is a vector point function.**

If $V = V_1\,\hat{\mathbf{i}} + V_2\,\hat{\mathbf{j}} + V_3\,\hat{\mathbf{k}}$, $\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$, then

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

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7.14.1 Irrotational Vector Field

If curl $\mathbf{V} = 0$ then, vector field $\mathbf{V}$ is called irrotational vector field.

A field which is not irrotational is called a **vortex field**.

NOTE:

A vector $\mathbf{F}$ is conservative if curl $\mathbf{F} = 0$.

7.14.2 Physical Interpretation of Curl

Here, we shall interpret curl in the context of a uniform rotating rigid body about an axis.

Let $\omega = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ be an angular velocity of a rigid body rotating about fixed point O.

The velocity $\mathbf{V}$ of any point $P(x, y, z)$ on the body is given by $\mathbf{V} = \omega \times \mathbf{r}$, where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector of P.

$$\therefore \quad V = \omega \times \mathbf{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

and

$$\text{curl}(\mathbf{V}) = \nabla \times \mathbf{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}$$

[$\because \omega_1, \omega_2, \omega_3$ are constants.]

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$\Rightarrow \quad \omega = \frac{1}{2} \text{curl } \mathbf{V}$$

Thus, the angular velocity at any point is equal to half the curl of the linear velocity

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Chapter 1

Complex Numbers and Their Geometrical Representation

$\sqrt{-4}$ $z = a + iy$

$\sqrt{-1} = i \Rightarrow \sqrt{-1 \times 4}$

$2i$

1/4

$\sqrt{-4} \Rightarrow \sqrt{-1 \times 4} = 2i$

1.1 Complex Numbers

A number of the form $x + iy$ where $i = \sqrt{(-1)}$ and x, y are both real number, is called a complex number. A complex number is also defined as an ordered pair (x, y) of real numbers. It is represented by $z = (x + iy)$ or (x, y) then x is called, **real part** and y is called the **imaginary part** of the complex number z *i.e.* $x = R(z)$ and $y = I(z)$.

Therefore in the complex number $z = a + ib$ we have

$z = 2 + 5i$

$$R(z) = \text{real part of } z = a$$

$$I(z) = \text{imaginary part of } z = b$$

NOTE:

1. A complex number is said to be purely real if its imaginary part is zero.
2. A complex number is said to be purely imaginary if its real part is zero.
3. The complex number $2 + 0\,i$ may be written as 2.
4. The set of complex number is denote by C.

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1.2 Algebra of Complex Numbers

1.2.1 Equality of Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$, or (x_1, y_1) and $z_2 = x_2 + iy_2$, or (x_2, y_2) are said to be equal if $x_1 = x_2$ and $y_1 = y_2$. Hence two complex numbers are equal if and only if the real part of one is equal to the real part of other and the imaginary part of one is equal to the imaginary part of the other.

1.2.2 Addition of Complex Numbers

If $z_1 = (x_1 + iy_1)$ or (x_1, y_1) and $z_2 = (x_2 + iy_2)$ or (x_2, y_2) are the two complex numbers, then the sum of z_1 and z_2 is written as $z_1 + z_2$ and defined as

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

or

$$(z_1 + z_2) = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

1.2.3 Properties of the Addition of Complex Numbers

Theorem 1: (Commutativity of addition in C).

To show that

$$z_1 + z_2 = z_2 + z_1$$

where z_1 and z_2 are any complex numbers. **[B.C.A. (Avadh) 2009, 07]**

Proof: Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

where x_1, x_2, y_1, y_2 are real numbers

Then
$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) \\ &= (x_1 + x_2, y_1 + y_2) && \text{[by addition in C]} \\ &= (x_2 + x_1, y_2 + y_1) && \text{[addition of real number is commutative]} \\ &= (x_2, y_2) + (x_1, y_1) \\ &= z_2 + z_1. \end{aligned}$$

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NOTE:

1. The complex number $(0, 0)$ or $0 + i0$ is the additive identity or zero element of complex number.

2. The complex number $(-a, -b)$ is the additive inverse of the complex number (a, b) since

$$(a, b) + (-a, -b) = (a - a, b - b) = (0, 0).$$

3. The complex number $(-a, -b)$ is called the negative of complex number (a, b) and we denote

$$(-a, -b) = -(a, b).$$

4. If z_1, z_2 be two complex number then subtraction of z_1 and z_2 is defined by

$$z - z_2 \text{ if } z_1 = (x_1, y_1) \text{ and } z_2 = (x_2, y_2)$$

$$\therefore \quad z_1 - z_2 = (x_1, y_1) - (x_2, y_2)$$

$$= (x_1 - x_2, y_1 - y_2).$$

(2x+3j)(

5. If u, v and w be any complex numbers then

$$u + w = v + w \Rightarrow u = v$$

cancellation law hold in addition in C. → Group of z^2

(2+5i) (4

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1.3 Division in C

A complex number (a, b) is said to be divisible by a complex number (c, d) if there exists a complex number (x, y) such that

$$(x, y)(c, d) = (a, b)$$

$$\Rightarrow \quad (xc - yd, xd + yc) = (a, b)$$

$$\Rightarrow \quad xc - yd = a \text{ and } xd + yc = b$$

Solve these equations we get

$$x = \frac{ac + bd}{c^2 + d^2}, \; y = \frac{bc - ad}{c^2 + d^2}$$

If z_1, z_2 be two complex number then division is $\frac{z_1}{z_2} = z_1(z_2)^{-1}$.

1.4 Symbol i and Its Powers

We denote the complex number $(0, 1)$ by i. Then

$$i^2 = (0,1)(0,1) = (0.0 - 1.1, 0.1 + 1.0) = (-1, 0)$$

$$\therefore \quad i^2 = (-1, 0)$$

$$\Rightarrow \quad i^3 = -i, i^4 = 1, i^6 = 1 \ldots\ldots$$

1.5 Conjugate of a Complex Number

The conjugate of complex number $z = x + iy$ is obtain by putting $i = -i$ and $-i = i$ it is denoted by $\bar{z} = x - iy$

Thus if $\quad z = -3 + 4i \Rightarrow \bar{z} = -3 - 4i$

The following results are for conjugate:

1. Two complex numbers are equal if and only if their conjugates are equal *i.e.*,

$$z_1 = z_2 \Leftrightarrow \overline{z_1} = \overline{z_2}$$

2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

   and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ provided $z_2 \neq 0$

3. $\overline{(\bar{z})} = z$

4. Let $z = x + iy, \bar{z} = x - iy \Rightarrow z + \bar{z} = 2x$

   Thus, sum of two conjugate complex numbers is a real number, equal to twice the real part of each.

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1.6 Separation of Real and Imaginary Part of Com[plex] Number

Let the complex number $= \dfrac{a+ib}{c+id}$

$$= \frac{(a+ib)(c-id)}{(c+id)(c-id)} \quad \text{[Multiplying } N_r \text{ and } D_r \text{ by conjugate}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \frac{(ac+bd)}{(c^2+d^2)} + i\frac{(bc-ad)}{(c^2+d^2)}$$

$$= A + iB \text{ where } A = \frac{ca+bd}{c^2+d^2}, B = \frac{bc-ad}{c^2+d^2}$$

A is real part and B is imaginary part of $\left(\dfrac{a+ib}{c+id}\right)$.

> **NOTE:**
>
> For separation of real and imaginary part of complex number multiplying N_r and D_r [by] conjugate of D_r.

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1.7 Modulus of a Complex Number

If $z=(x, y)$ or $z=(x+iy)$ be any complex number, then the non-negative real num[ber] $\sqrt{x^2+y^2}$ is called the modulus or absolute value of complex number z and it is deno[ted] by $|z|$ or mod (z). Therefore

$$|2+3i| = \sqrt{(2)^2+(3)^2} = \sqrt{4+9} = \sqrt{13}$$

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Theorem 5: The modulus of the product of two complex numbers is the product of their moduli.

[B.C.A. (Meerut) 2006]

Proof: We have $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2})$

$$= (z_1 z_2)(\overline{z_1}\,\overline{z_2})$$

$$= (z_1 \overline{z_1})(z_2 \overline{z_2})$$

$$= |z_1|^2 |z_2|^2$$

So that $\qquad |z_1 z_2|^2 = |z_1|^2 |z_2|^2$

or $\qquad |z_1 z_2| = |z_1||z_2|$

In general, $\quad |z_1 z_2 \ldots\ldots z_n| = |z_1||z_2|\ldots\ldots|z_n|$.

Theorem 6: The modulus of the sum of two complex numbers is always less than or equal to the sum of their moduli

or $\qquad |z_1 + z_2| \le |z_1| + |z_2|$. **[B.C.A. (Kanpur) 2011, 07; B.C.A. (Meerut) 2008]**

Proof: We have to show that

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Now $\qquad |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$

$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$

$$= (z_1\overline{z_1} + z_2\overline{z_2}) + (z_1\overline{z_2} + \overline{z_1}z_2)$$

$$= |z_1|^2 + |z_2|^2 + 2R(z_1\overline{z_2})$$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}| \qquad [\because z_1\overline{z_1} = |z_1|]$$

Also $\qquad z_1\overline{z_2} + \overline{z_1}z_2 = 2R(z_1\overline{z_1}) \le 2|z_1\overline{z_2}|$

$\therefore \qquad |z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}|$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \qquad [|\overline{z_2}| = |z_2|]$$

$$|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$$

$\Rightarrow \qquad |z_1 + z_2| \le |z_1| + |z_2|$.

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Example 1: *Express $\frac{2+3i}{4+5i}$ in the form $x+iy$.*

Solution: Multiplying the numerator and denominator of the given fraction by the conjugate complex of the denominator, we have

$$\Rightarrow \quad \frac{2+3i}{4+5i}=\frac{(2+3i)}{(4+5i)}\times\frac{(4-5i)}{(4-5i)}=\frac{8-10i+12i-15i^2}{16-25i^2}$$

$$=\frac{23+2i}{16+25}=\frac{23+2i}{41}=\frac{23}{41}+\frac{2}{41}i$$

$\because$ the real part $x=\frac{23}{41}$ and the imaginary part $y=\frac{2}{41}$.

H.4

Example 2: *Find real numbers x and y, if*

$$x+iy=\frac{2-3i}{4+7i}.$$

Solution: We have

$$\frac{2-3i}{4+7i}=\frac{(2-3i)(4-7i)}{(4+7i)(4-7i)}=\frac{8-14i-12i+21i^2}{16-49i^2}=\frac{(8-21)-26i}{16+49}$$

$$=\frac{-13-26i}{65}=\frac{-13}{65}-\frac{26}{65}i=-\frac{1}{5}-\frac{2}{5}i$$

H.4

$$x+iy=-\frac{1}{5}-\frac{2i}{5}\Rightarrow x=-1/5,\ y=-2/5.$$

Example 3: *Express $1-i$ in the modulus amplitude form.*

Solution: Let $1-i=r(\cos\theta+i\sin\theta)$

$\Rightarrow$ Equating real and imaginary part on both side,

$$\Rightarrow \quad r\cos\theta=1 \qquad ...(1)$$

$$\Rightarrow \quad r\sin\theta=-1 \qquad ...(2)$$

$\Rightarrow$ Squaring and adding (1) and (2), we have

$$1^2=1+1=2, \quad \therefore \quad r=+\sqrt{2}$$

$\Rightarrow$ Divided (2) by (1)

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$\Rightarrow$ $$\tan\theta = -1$$

$\Rightarrow$ $$\theta = -\frac{\pi}{4}$$

$\Rightarrow$ Hence, $$1 - i = \sqrt{2}\,\{\cos(-\pi/4) + i\sin(-\pi/4)\}.$$

Example 4: *Reduce to the form $r(\cos\theta + i\sin\theta)$ the quantity $(-1 + i\sqrt{3})$.*

[B.C.A. (Avadh) 2007]

Solution: Let $-1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ Equating real and imaginary parts, we have

$\Rightarrow$ $$r\cos\theta = -1 \quad ...(1)$$

$\Rightarrow$ $$r\sin\theta = \sqrt{3} \quad ...(2)$$

$\Rightarrow$ Squaring and added (1) and (2), we have

$\Rightarrow$ $$r^2 = 1 + 3 = 4; \qquad \therefore\ r = 2$$

$\Rightarrow$ Dividing (2) by (1), we have $\tan\theta = -\sqrt{3}$

$\Rightarrow$ $$\theta = \frac{2\pi}{3}$$

$\Rightarrow$ Choosing the values of θ lying between $-\pi$ and π for which $\cos\theta$ is negative and $\sin\theta$ is positive.

Hence $$-1 + i\sqrt{3} = 2\,[\cos(2\pi/3) + i\sin(2\pi/3)].$$

Example 5: *Express $\dfrac{(\sqrt{3} - 1) + i(\sqrt{3} + 1)}{2\sqrt{2}}$ in the Trigonometric form.*

Solution: Let $\dfrac{(\sqrt{3} - 1)}{2\sqrt{2}} + i\dfrac{(\sqrt{3} + 1)}{2\sqrt{2}} = r(\cos\theta + i\sin\theta)$

Equating real and imaginary part, we have

$$r\cos\theta = \frac{\sqrt{3} - 1}{2\sqrt{2}} \quad ...(1)$$

$$r\sin\theta = \frac{\sqrt{3} + 1}{2\sqrt{2}} \quad ...(2)$$

Squaring (1), (2) and adding these, we get

$$r^2 = \frac{(\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2}{8} = 1 \text{ i.e., } r = 1$$

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Example 6: If $x + iy = \dfrac{3}{2 + \cos\theta + i\sin\theta}$, *prove that*

$$(x-1)(x-3) + y^2 = 0.$$

[B.C.A. (Bhopal) 2012]

Solution: We have

$$x + iy = \frac{3}{2 + \cos\theta + i\sin\theta}$$

$$\Rightarrow \quad x + iy = \frac{3(2 + \cos\theta - i\sin\theta)}{[(2+\cos\theta) + i\sin\theta][(2+\cos\theta) - i\sin\theta]}$$

$$\Rightarrow \quad x + iy = \frac{3(2 + \cos\theta - i\sin\theta)}{(2+\cos\theta)^2 + \sin^2\theta}$$

$$\Rightarrow \quad x + iy = \frac{6 + 3\cos\theta - i3\sin\theta}{5 + 4\cos\theta}$$

Equating real and imaginary part on both side.

$$x = \frac{6 + 3\cos\theta}{5 + 4\cos\theta}, \quad y = \frac{-3\sin\theta}{5 + 4\cos\theta}$$

$$\therefore \quad (x-1)(x-3) = \left(\frac{6 + 3\cos\theta}{5 + 4\cos\theta} - 1\right)\left(\frac{6 + 3\cos\theta}{5 + 4\cos\theta} - 3\right)$$

$$= \left(\frac{1 - \cos\theta}{5 + 4\cos\theta}\right)\left(\frac{-9 - 9\cos\theta}{5 + 4\cos\theta}\right)$$

$$= -9\,\frac{(1 - \cos\theta)(1 + \cos\theta)}{(5 + 4\cos\theta)^2}$$

$$= -\frac{9\sin^2\theta}{(5 + 4\cos\theta)^2} \qquad \text{...(1)}$$

Also

$$y^2 = \frac{9\sin^2\theta}{(5 + 4\cos\theta)^2} \qquad \text{...(2)}$$

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For example: $\sin^{-1}(1/2) = \pi/6$ while $\mathrm{Sin}^{-1}(1/2) = n\pi + (-1)^n \frac{\pi}{6}$

where n is any integer.

$$\therefore \qquad \mathrm{Sin}^{-1} x = n\pi + (-1)^n \sin^{-1} x$$

Similarly, we can write the relation in general value and principal value for other circul functions.

$$\mathrm{Cos}^{-1}(x) = 2n\pi \pm \cos^{-1} x$$

$$\mathrm{Tan}^{-1} x = n\pi + \tan^{-1} x$$

$$\mathrm{Cosec}^{-1} x = n\pi + (-1)^n \operatorname{cosec}^{-1} x$$

$$\mathrm{Sec}^{-1} x = 2n\pi \pm \sec^{-1} x$$

$$\mathrm{Cot}^{-1} x = n\pi + \cot^{-1} x$$

where n is any integer, positive or zero.

3.3 Relation between Inverse Functions

1. We have $\sin^{-1} x = \operatorname{cosec}^{-1}(1/x)$, $\cos^{-1} x = \sec^{-1}(1/x)$ and $\tan^{-1} x = \cot^{-1}(1/x)$

We know $\sin\theta = x \Rightarrow \theta = \sin^{-1} x$

and if $\sin\theta = x$, then $\operatorname{cosec}\theta = 1/x \Rightarrow \theta = \operatorname{cosec}^{-1}(1/x)$

$$\therefore \qquad \sin^{-1} x = \operatorname{cosec}^{-1}(1/x)$$

Similarly, we can show that $\cos^{-1}(x) = \sec^{-1}(1/x)$ and $\cot^{-1}(x) = \tan^{-1}(1/x)$

2. From the definition of an inverse circular function,

we have $\theta = \sin^{-1}(\sin\theta)$ and $x = \sin(\sin^{-1} x)$

For, if $\sin\theta = x \Rightarrow \theta = \sin^{-1} x = \sin^{-1}(\sin\theta)$

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Similarly, $\theta = \cos^{-1}(\cos\theta)$ and $x = \cos(\cos^{-1} x)$

$\theta = \tan^{-1}(\tan\theta)$ and $x = \tan(\tan^{-1} x)$ etc.

3. To express one inverse circular function in terms of the others.

If $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

Then $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

i.e., $\cos\theta = \sqrt{1-x^2} \Rightarrow \theta = \cos^{-1}\sqrt{1-x^2}$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{x}{\sqrt{1-x^2}} \Rightarrow \theta = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

Hence, $\theta = \sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left\{\dfrac{x}{\sqrt{1-x^2}}\right\}$

Also by principle of reciprocity, we have

$$\theta = \text{cosec}^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Hence, $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left\{\dfrac{x}{\sqrt{1-x^2}}\right\}$

$$= \text{cosec}^{-1}(1/x) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Similarly, $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left\{\dfrac{\sqrt{1-x^2}}{x}\right\}$

and $\tan^{-1} x = \sin^{-1}\left\{\dfrac{x}{\sqrt{1+x^2}}\right\} = \cos^{-1}\left\{\dfrac{1}{\sqrt{1+x^2}}\right\}$ etc.

4. To show that

(i) $\sin^{-1}(-x) = -\sin^{-1} x$ **(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$**

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$.

Proof: (i) Put $-x = \sin\theta \Rightarrow \theta = \sin^{-1}(-x)$ ...(1)

Now, $-x = \sin\theta \Rightarrow x = -\sin\theta = \sin(-\theta)$

$\therefore$ $-\theta = \sin^{-1} x$ or $\theta = -\sin^{-1} x$ ...(2)

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2. To prove that

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

[B.C.A. (I.G.N.O.U.) 2008]

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(iv) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$.

[B.C.A. (Bhopal) 2011, 06, 04]

Proof: (i) We know

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\therefore \quad A + B = \tan^{-1}\left\{\frac{\tan A + \tan B}{1 - \tan A \tan B}\right\} \quad \text{...(1)}$$

Let $A = \tan^{-1} x$ and $B = \tan^{-1} y$

$\Rightarrow$ $x = \tan A$ and $y = \tan B$

Put these values in (1), we get

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left\{\frac{x+y}{1-xy}\right\}$$

(ii) We know $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\therefore \quad A - B = \tan^{-1}\left\{\frac{\tan A - \tan B}{1 + \tan A \tan B}\right\} \quad \text{...(2)}$$

Let $A = \tan^{-1} x \Rightarrow x = \tan A$ and $B = \tan^{-1} y \Rightarrow y = \tan B$

Put these values in (2), we get

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left\{\frac{x-y}{1+xy}\right\}.$$

(iii) Put $x = y$ in (1), we get

$$\tan^{-1} x + \tan^{-1} x = \tan^{-1}\left(\frac{x+x}{1-x.x}\right)$$

$$\Rightarrow \quad 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

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Example 1: *Prove that* $\tan^{-1}(1/3)+\tan^{-1}(1/5)+\tan^{-1}(1/7)+\tan^{-1}(1/8)=\pi/4.$

[B.C.A. (Meerut) 2000]

Solution: Hence,

$$\text{L.H.S.} = \tan^{-1}(1/3)+\tan^{-1}(1/5)+\tan^{-1}(1/7)+\tan^{-1}(1/8)$$

$$= \tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}.\frac{1}{5}} + \tan^{-1}\frac{1/7+1/8}{1-1/7.1/8}, \text{ by the formula}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}4/7+\tan^{-1}3/11$$

$$= \tan^{-1}\left(\frac{4/7+3/11}{1-4/7.3/11}\right)=\tan^{-1}1=\pi/4=\text{R.H.S.}$$

Example 2: *Prove that* $\sin^{-1}4/5+\sin^{-1}5/13+\sin^{-1}16/65=\pi/2.$

[B.C.A. (I.G.N.O.U) 2010]

Solution: Hence,

$$\text{L.H.S.} = \sin^{-1}4/5+\sin^{-1}5/13+\sin^{-1}16/65$$

$$= (\sin^{-1}4/5+\sin^{-1}5/13)+\sin^{-1}16/65$$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{\left(1-\left(\frac{5}{13}\right)^2\right)}+5/13\sqrt{1-\left(\frac{4}{5}\right)^2}\right]+\sin^{-1}16/65$$

$$[\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}\}]$$

$$= \sin^{-1}\left[\frac{4}{5}.\frac{12}{13}+\frac{5}{13}.\frac{3}{5}\right]+\sin^{-1}16/65$$

$$= \sin^{-1}\left(\frac{63}{65}\right)+\sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right)+\cos^{-1}\sqrt{\left\{1-\left(\frac{16}{65}\right)^2\right\}} \qquad [\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}]$$

$$= \sin^{-1}\left(\frac{63}{65}\right)+\cos^{-1}\left(\frac{63}{65}\right)=\pi/2, \text{ for } \sin^{-1}x+\cos^{-1}x=\pi/2=\text{R.H.S.}$$

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Example 5: ***If*** $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, ***prove that***

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

[B.C.A. (Meerut) 2004]

Solution: We know that,

$$\cos^{-1} x + \cos^{-1} y = \cos^{-1} [xy - \sqrt{1-x^2} . \sqrt{1-y^2}].$$

$\Rightarrow$ $$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$$

$\Rightarrow$ $$\cos^{-1} [xy - \sqrt{1-x^2} . \sqrt{1-y^2}] = \pi - \cos^{-1} z$$

$\Rightarrow$ $$[xy - \sqrt{1-x^2} . \sqrt{1-y^2}] = \cos (\pi - \cos^{-1} z)$$

$\Rightarrow$ $$[xy - \sqrt{1-x^2} . \sqrt{1-y^2}] = - \cos \cos^{-1} z$$

$\Rightarrow$ $$[xy - \sqrt{1-x^2} . \sqrt{1-y^2}] = - z$$

$\Rightarrow$ $$xy + z = \sqrt{1-x^2} . \sqrt{1-y^2}$$

Taking square on both sides

$\Rightarrow$ $$(xy + z)^2 = (1 - x^2)(1 - y^2).$$

$\Rightarrow$ $$x^2 y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2 y^2$$

$\Rightarrow$ $x^2 + y^2 + 2xyz = 1$ **Proved.**

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Differential Equations of First Order and First Degree

9.1 Differential Equations

Definition: "*A **Differential Equation** is an equation that involves independent and dependent variables and the derivatives of the dependent variables*."

9.1.1 Ordinary Differential Equation

A differential equation which involves only one independent variable is an **Ordinary Differential Equation.** Thus, the differential equations:

$$\sin y \, dy = \cos x \, dx \qquad \text{...(1)}$$

$$\frac{d^2 y}{dx^2} = a^2 y \qquad \text{...(2)}$$

$$x^3 \left(\frac{d^2 y}{dx^2}\right)^3 + y^2 \left(\frac{dy}{dx}\right)^4 + y^3 = 0 \qquad \text{...(3)}$$

$$\frac{d^2 y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0 \qquad \text{...(4)}$$

$$\frac{[1 + (dy/dx)^2]^{3/2}}{d^2 y / dx^2} = \rho \qquad \text{...(5)}$$

are all examples of ordinary differential equations. They involve single independent variable x.

Solved Examples

Example 1: *Show that $y = c_1 \cos(\log x) + c_2 \sin(\log x)$ is a solution of the differential equation*

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

[B.C.A. (Agra) 2010, 03, 02]

Solution: We have $y = c_1 \cos(\log x) + c_2 \sin(\log x)$ ...(1)

Differential (1) w.r.t (x), we get

$$\frac{dy}{dx} = c_1 \{-\sin(\log x)\}\frac{1}{x} + c_2 \cos(\log x) \cdot \frac{1}{x}$$

or
$$x\frac{dy}{dx} = -c_1 \sin(\log x) + c_2 \cos(\log x) \quad ...(2)$$

Again differentiate (2) w.r.t (x), we get

$$\frac{dy}{dx} + x\frac{d^2 y}{dx^2} = -\frac{c_1 \cos(\log x)}{x} - \frac{c_2 \sin(\log x)}{x}$$

or
$$x\frac{dy}{dx} + x^2\frac{d^2 y}{dx^2} = -\{c_1 \cos(\log x) + c_2 \sin(\log x)\}$$

or
$$x\frac{dy}{dx} + x^2\frac{d^2 y}{dx^2} = -y \quad \text{[From (1)]}$$

or
$$x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Hence, $y = c_1 \cos(\log x) + c_2 \sin(\log x)$

be a solution of the given differential equation.

Example 2: *Form the differential equation from $y = Ae^{2x} + Be^{x} + C$, where A, B and C are constant.*

[B.C.A. (Bundelkhand) 2011, 09, 06]

Solution: The given equation is

$$y = Ae^{2x} + Be^{x} + C \quad ...(1)$$

Differentiating (1) w.r.t to x, we get

$$\frac{dy}{dx} = 2Ae^{2x} + Be^{x}$$

$$\Rightarrow \quad e^{-x}\frac{dy}{dx} = 2Ae^{x} + B \quad ...(2)$$

9.4 Separation of Variables

We start with procedures for solving first order differential equations whose variables can be separated. This method is called **separation of variables** and is outlined as follows.

Let the first order differential equation be

$$\phi\left(x, y \frac{dy}{dx}\right) = 0 \qquad ...(1)$$

Solving the differential equation (1) for $\frac{dy}{dx}$ we get, say,

$$\frac{dy}{dx} = f(x, y) \qquad ...(2)$$

Suppose that $f(x, y)$ can be written in the following form

$$f(x, y) = f_1(x) / f_2(y), \qquad ...(3)$$

where f_1 and f_2 are continuous, then (2) can be written with variables separated in the differential form:

$$f_2(y)\frac{dy}{dx} = f_1(x) \quad \text{or} \quad f_2(y)\,dy = f_1(x)\,dx \qquad ...(4)$$

Thus, a differential equation which can be represented in the form (3) is called **variables separable form.**

Now integrating both sides of (4), we get

$$\int f_2(y)\, f(y)\, dy = \int f_1(x)\, dx + c \qquad ...(5)$$

where c is an arbitrary constant of integration. This constant, being arbitrary, can be put in any form and either side of (5) as we like. Since the integral involved in (5) can be evaluated hence the result will be free from any differential and there by giving the general solution of the differential equation (2).

Example 10: ***Solve*** $(x + 1)\frac{dy}{dx} = x(y^2 + 1)$.

[B.C.A. (Agra) 2012, 08, 06, 02]

Solution: We have $(x + 1)\frac{dy}{dx} = x(y^2 + 1)$

Separate the variables, we get

$$\frac{dy}{1 + y^2} = \left(\frac{x}{x + 1}\right)dx$$

Integrate both sides and add constant of integration

$$\int \frac{dy}{1 + y^2} = \int \left(1 - \frac{1}{x + 1}\right)dx + c$$

$$\tan^{-1}(y) = x - \log(1 + x) + c$$

$$\log(1 + x) = c + x - \tan^{-1} y$$

$$(1 + x) = e^{c + x - \tan^{-1} y}$$

$$(1 + x) = e^{c}\, e^{x - \tan^{-1} y}$$

$$(1 + x) = \lambda\, e^{x - \tan^{-1} y}. \qquad [\text{where } \lambda = e^{c} = \text{constant}]$$

Example 11: ***Solve*** $x(e^{y} + 4)\,dx + e^{x+y}\,dy = 0.$ **[B.C.A. (Kurukshetra) 2010]**

Solution: We have

$$x(e^{y} + 4)\,dx + e^{x+y}\,dy = 0$$

Separate the variables, we get

$$\left(\frac{x}{e^{x}}\right) dx + \left(\frac{e^{y}}{e^{y} + 4}\right) dy = 0$$

Integrate we obtain

$$\int xe^{-x}\,dx + \int \frac{e^{y}}{e^{y} + 4}\,dy = c$$

or $$\boxed{-xe^{-x} - e^{-x} + \log(e^{y} + 4) = c}$$

$\frac{x+1-1}{x+1}$

or $$\log(e^{y} + 4) - (x + 1)\,e^{-x} = c.$$

9.5 Homogeneous Function

Definition: *"The function given by $z = f(x, y)$ is said to be homogeneous function of degree n if $f(tx, ty) = t^n f(x, y)$."*

9.5.1 Homogeneous Differential Equations

Definition: *"A differential equation of the form*

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} \quad ...(1)$$

where $f_1(x, y)$, $f_2(x, y)$ are homogeneous functions of the same degree, n (say) in x and y, is called ***homogeneous differential equation."***

Changing a Homogeneous Differential Equation to Variables Separable Form: The differential equation (1) can be transformed into an equation whose variables are separable by letting $v = y/x$, then

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}. \quad ...(2)$$

Put this value in (1), we get

$$v + x\frac{dv}{dx} = f(v) \quad ...(3)$$

Because $f_1(x, y)$ and $f_2(x, y)$ both are homogeneous functions of the same degree (say n)
So that

$$\frac{f_1(x, y)}{f_2(x, y)} = \frac{x^n f_1(y/x)}{x^n f_2(y/x)} = \frac{f_1(v)}{f_2(v)} = f(v).$$

on separating the variables, (3) can be written as

$$\frac{dv}{f(v) - v} = \frac{dx}{x} \quad ...(4)$$

Integrating both sides of (4), we obtain a solution in v and x as follows:

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c.$$

Replacing v by y/x in the above equation, we get the general solution of the differential equation (1).

Example 29: *Solve $x^2 dy + y(x + y) dx = 0$.* **[B.C.A. (Rohilkhand) 2012, 07]**

Solution: The given differential equation can be written as

$$\frac{dy}{dx} + \frac{y(x + y)}{x^2} = 0 \quad ...(1)$$

which is a homogeneous differential equation.

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

(1) becomes $v + x\dfrac{dv}{dx} + \dfrac{vx(x + vx)}{x^2} = 0,$

or $v + x\dfrac{dv}{dx} + v(1 + v) = 0$ or $x\dfrac{dv}{dx} + v^2 + 2v = 0$

or $\dfrac{dv}{(v^2 + 2v)} + \dfrac{dx}{x} = 0$ or $\dfrac{1}{2}\left(\dfrac{1}{v} - \dfrac{1}{v + 2}\right)dv + \dfrac{1}{x}dx = 0$

which is a differential equation in variables separable form. Hence on integration, we get

$$\frac{1}{2}[\log v - \log(v + 2)] + \log x = C$$

or $\log x\sqrt{\dfrac{v}{v + 2}} = C$ or $\log x\sqrt{\dfrac{y/x}{(y/x) + 2}} = C$ $[\because v = y/x]$

or $x\sqrt{\dfrac{y}{y + 2x}} = e^C = c'$, say

or $x^2 y = c(y + 2x),$

c being an arbitrary constant $c = c'^2$ which is the required solution.

9.6 Non-homogeneous Equations of the First Degree in x and y

Equations Reducible to a Homogeneous Form

Definition: "*Equations of the form*

$$\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C} \qquad ...(1)$$

is known as ***differential equations reducible to homogeneous form.***"

Case I: When $(a/A)\neq(b/B)$.

In this case, (1) can be reduced to homogeneous form by the following substitution which change variables x and y to new variables X and Y,

$$x=X+h,\ y=Y+k \qquad ...(2)$$

where h and k are arbitrary constants and they can be so chosen that the new differential equation may be homogeneous. Now (2) gives

$$dx=dX \quad \text{and} \quad dy=dY \qquad ...(3)$$

Thus, (1) reduces to

$$\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C}$$

or

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)} \qquad ...(4)$$

Now, choose h and k such that

$$ah+bk+c=0 \qquad ...(5)$$

and

$$Ah+Bk+C=0 \qquad ...(6)$$

Example 37. Solve $(2x + y - 3)\,dy = (x + 2y - 3)\,dx$. [B.C.A. (Lucknow) 201_]

Solution. The given differential equation

$$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3} \qquad (1)$$

is reducible to homogeneous form. Here $(a/A) \neq (b/B)$, that is $\left(\frac{1}{2}\right) \neq \left(\frac{2}{1}\right)$. Hence, putting $x = X + h$ and $y = Y + k$ the given differential equation reduces to the form

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)} \qquad (2)$$

Now, choose h and k such that

$$h + 2k - 3 = 0; \; 2h + k - 3 = 0.$$

Solving these equations, we get

$$\frac{h}{-6+3} = \frac{k}{-6+3} = \frac{1}{1-4}$$

$$h = 1, \; k = 1.$$

$\frac{v+3-1+1}{v+2}$

$v+2$

Substituting these values in (1), we obtain

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \qquad (3)$$

which is a differential equation in homogeneous form in X and Y, Y being dependent variable. Hence, putting

$$Y = vX \qquad \text{and} \qquad \frac{dY}{dX} = v + X\frac{dv}{dX}$$

(3) transforms in the form

$$v + X\frac{dv}{dX} = \frac{X + 2vX}{2X + vX}$$

or $$X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v} \qquad \text{or} \qquad \frac{2+v}{1-v^2}\,dv = \frac{dX}{X}$$

which is in variables separable form in v and X, hence on integration, we get

$$\int \frac{2+v}{1-v^2}\,dv = \int \frac{dX}{X} + C$$

or $$\frac{1}{2}\int \left[\frac{3}{1-v} + \frac{1}{1+v}\right] dv = \log X + C$$

9.7 Linear Differential Equations

A differential equation is called **linear** if the dependent variable y and its derivatives with respect to independent variable x occur in the first degree only and there is no restriction of any kind on the occurrence of the independent variable x.

Definition: "*A differential equation of the form*

$$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

Working Rule

To solve a linear differential equation of the first order, students should remember the following points:

1. Arrange the given differential equation in the standard form.

$$\frac{dy}{dx} + Py = Q \qquad \text{or} \qquad \frac{dx}{dy} + Px = Q$$

2. Write down its I.F. $= e^{\int P\,dx}$ or $e^{\int P\,dy}$ and evaluate it.

3. The general solution of the differential equation will be written as

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$$

or

$$xe^{\int P\,dy} = \int Qe^{\int P\,dy}\,dy + c$$

c being arbitrary constant of integration.

Example 42: *Solve* $(1 + x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$. [B.C.A. (Garhwal) 2007, 03]

Solution: The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{(1+x^2)}y = \frac{4x^2}{(1+x^2)}$$

which is a linear differential equation and its of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = \frac{2x}{(1+x^2)} \quad \text{and } Q = \frac{4x^2}{(1+x^2)}$$

$\therefore$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{(1+x^2)}dx} = e^{\log(1+x^2)} = (1+x^2).$$

Hence, its solution is

$$y \times (\text{I.F.}) = \int \{Q \times (\text{I.F.})\}\,dx + c$$

i.e.,

$$y(1+x^2) = \int \left\{\frac{4x^2}{(1+x^2)} \cdot (1+x^2)\right\} dx + c,$$

or

$$y(1+x^2) = \left(\frac{4}{3}\right)x^3 + c \qquad \text{or} \qquad 3y(1+x^2) - 4x^3 = 3c$$

which is the required solution.

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linear form.

Example 53: *Solve* $(1-x^2)\frac{dy}{dx}+xy=xy^2$. [B.C.A. (Rohilkhand) 2003, 01]

Solution: Dividing throughout the given differential equation by $y^2(1-x^2)$, we get

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{x}{1-x^2}\cdot\frac{1}{y}=x \quad ...(1)$$

Putting $\frac{1}{y}=v$, so that $-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$ in equation (1), we get

$$-\frac{dv}{dx}+\frac{x}{1-x^2}v=x \quad \text{or} \quad \frac{dv}{dx}+\frac{x}{x^2-1}v=-x \quad ...(2)$$

which is linear differential equation in v.

Here $P=\frac{x}{x^2-1}$ and $Q=-x$

$\therefore$ Integrating factor (I.F.) $=e^{\int P\,dx}=e^{\int \frac{x}{x^2-1}dx}\; e^{\frac{1}{2}\log(x^2-1)}=\sqrt{x^2-1}$.

Multiplying both sides of equation (2) by integrating factor (I.F.) and integrating, we get

$$v\cdot\sqrt{x^2-1}=c+\int(-x)\cdot\sqrt{x^2-1}\,dx$$

$$=c-\int t^2\,dt,=c-\frac{1}{3}t^3 \quad \text{(on putting } x^2-1=t^2\text{)}$$

or

$$\frac{1}{y}\cdot\sqrt{x^2-1}=c-\frac{1}{3}(x^2-1)^{3/2}$$

or

$$\sqrt{x^2-1}=y\left[c-\frac{1}{3}(x^2-1)^{3/2}\right].$$

This is the required solution.

Example 54: *Solve* $\frac{dy}{dx}+\frac{y}{x}=y^2\sin x$

9.9 Exact Differential Equations

The differential equation of the form $M(x, y)\, dx + N(x, y)\, dy = 0$ is called exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

where $\frac{\partial M}{\partial y}$ = partial differentiation of M w.r.t. y when x as constant

and $\frac{\partial N}{\partial x}$ = partial differentiation of N w.r.t. x when y as constant.

Working Rule

1. First integrate M with respect to x regarding y as a constant.
2. Integrate N with respect to y, keeping x constant, and retaining only those terms which have not been already obtained by the integration of M.
3. The sum of the expressions, thus obtained equated to an arbitrary constant will be the required solution.

Example 68: *Solve the equation* $(1 + 4xy + 2y^2)\, dx + (1 + 4xy + 2x^2)\, dy = 0$.

Solution: Here $M = 1 + 4xy + 2y^2$ and $N = 1 + 4xy + 2x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 4x + 4y \quad \text{and} \quad \frac{\partial N}{\partial x} = 4y + 4x$$

Hence
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

it follows that the given differential equation is exact.

Now,
$$\int_{y\text{-constant}} M\, dx = \int (1 + 4xy + 2y^2)\, dx = x + 2x^2 y + 2xy^2$$

and
$$\int_{x\text{-constant}} N\, dy = \int (1 + 4xy + 2x^2)\, dy = y + 2xy^2 + 2x^2 y$$

Again, the only new term obtained on integrating N with respect to y is y, as the terms $2xy^2 + 2x^2 y$ are already present in the integrating of M.

Hence, the solution of the given differential equation is

$$(x + 2x^2 y + 2xy^2) + y = c \quad \text{or} \quad x + y + 2xy(x + y) = c$$

or
$$(x + y)(1 + 2xy) = c.$$

Example 72: *Solve* $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

Solution: Here $M = (1 + e^{x/y})$; $N = e^{x/y}\left(1 - \frac{x}{y}\right)$.

Therefore, $$\frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2}e^{x/y}$$

and $$\frac{\partial N}{\partial x} = e^{x/y}\cdot\left(0 - \frac{1}{y}\right) + \left(1 - \frac{x}{y}\right)\cdot\left(e^{x/y}\cdot\frac{1}{y}\right) = -\frac{x}{y^2}e^{x/y}.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, it follows that the given differential equation is exact.

Now, $$\int_{y\text{-constant}} M\,dx = \int (1 + e^{x/y})\,dx = x + ye^{x/y}$$

and $$\int_{y\text{-constant}} N\,dy = e^{x/y}\left(1 - \frac{x}{y}\right)dy;\ \text{put } y = \frac{1}{t}$$

$$= \int e^{xt}(1 - xt)\cdot\left(-\frac{dt}{t^2}\right) = -\int e^{xt}\frac{dt}{t^2} + \int xe^{xt}\cdot\frac{1}{t}\,dt$$

$$= -\int \frac{e^{xt}}{t^2}\,dt + \left\{e^{xt}\cdot\frac{1}{t} - \int e^{xt}\left(-\frac{1}{t^2}\right)dt\right\}, \text{ on integrating by parts}$$

$$= e^{xt/t}, \text{ as the two integrals cancel each other} = ye^{x/y}, \text{ as } t = 1/y.$$

The term $ye^{x/y}$ has already occurred in the integration of M. Therefore, no new term is obtained by integrating N with respect to y.

Hence, the required solution is

$$x + ye^{x/y} = c, \ c \text{ being arbitrary constant.}$$

Example 73: *Solve* $(e^y + 1)\cos x\,dx \;\; e^y \sin x\,dy = 0.$ **[B.C.A. (Kurukshetra) 2007]**

9.11 Rules for Finding Integrating Factors

Rule 1: When $Mx + Ny \neq 0$, and the equation is homogeneous, then an integrating factor of the differential equation

$$M\,dx + N\,dy = 0 \text{ is } \frac{1}{Mx + Ny}.$$

Rule 2: If the equation $M\,dx + N\,dy = 0$ has the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0, \quad \text{and} \quad Mx - Ny \neq 0,$$

an integrating factor is $\dfrac{1}{Mx - Ny}$.

Rule 3: If in the equation $M\,dx + N\,dy = 0$,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$$

(a function of x alone), then $e^{\int f(x)\,dx}$ is an integrating factor.

Example 80: *Solve* $(x^2 y - 2xy^2)\,dx - (x^3 - 3x^2 y)\,dy = 0.$ **[B.C.A. (Kanpur) 2007, 04]**

Solution: The given differential equation is homogeneous.

Also,
$$Mx + Ny = (x^3 y - 2x^2 y^2) - (x^3 y - 3x^2 y^2) = x^2 y^2 \neq 0$$

$\therefore$ Integrating factor (I.F.) $= \dfrac{1}{Mx + Ny} = \dfrac{1}{x^2 y^2}.$

Hence, multiplying the given differential equation by this integrating factor, we get

$$\left(\frac{1}{y} - 2x\right) dx + \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 0 \qquad \ldots(1)$$

Clearly for equation (1), $\dfrac{\partial M}{\partial y} = -\dfrac{1}{y^2} = \dfrac{\partial N}{\partial x}.$

Therefore, equation (1) is an exact differential equation.

Now,
$$\int_{y\text{-constant}} M\,dx = \int_{y\text{-constant}} \left(\frac{1}{y} - 2x\right) dx = \frac{x}{y} - x^2$$

and
$$\int_{x\text{-constant}} N\,dy = \int_{x\text{-constant}} \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 3 \log y + \frac{x}{y}.$$

The only new term in the integration of N is $3 \log y$, hence the required solution of the given differential equation is

$$\frac{x}{y} - x^2 + 3 \log y = c.$$

... $dx + 2y\,dy = 0.$ [B.C.A. (Agra) 2003]

Differential Equations of Second Order with Constant Coefficients

10.1 Definition

[B.C.A. (Agra) 2011, 09]

We have already defined a linear differential equation of the first order in chapter 9. If a differential equation in variables x and y is such that the dependent variable y and its derivatives occur only in the first degree and there occur no term containing product of derivatives of different order and the product of the derivative and the dependent variable, then such an equation is called a **linear differential equation** (of higher order). Also, the coefficients of the derivatives and the dependent variable y are constants, then it is called a **linear differential equation** (of higher order) **with constant coefficients.** [B.C.A. (Kanpur) 2008, 06]

Thus, a linear differential equation of nth order of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = Q$$

or

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = Q \quad \ldots(1)$$

where $a_1, a_2, \ldots, a_{n-1}, a_n$ are all constants and Q is some function of x, is called a **linear differential equation with constant coefficients.** [B.C.A. (Avadh) 2010, 08]

If $Q = 0$, then the equation (1) is called **homogeneous differential equation**. If $Q \neq 0$, then the equation (1) is called **non-homogeneous differential equation**.

[B.C.A. (Bundelkhand) 2008]

10.2 Auxiliary Equation

Let us consider the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = 0 \quad \dots(1)$$

Let $y = e^{mx}$ be a solution of equation (1), then by actual substitution, we have

$$e^{mx}\,[m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n] = 0$$

Hence, e^{mx} will be a solution of (1) if m is a root of the algebraic equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0 \quad \dots(2)$$

This equation is called the **auxiliary equation** or **characteristic equation** of the given differential equation (1).

In order to solve (1), we write the auxiliary equation (2) and solve it for m. Which gives different cases, according as the roots of the auxiliary equation (2) are real and distinct, real and repeated or complex.

NOTE:

It is worthwhile to note that the auxiliary equation can be written down from (1) by replacing y by 1, $\frac{dy}{dx}$ by m, $\frac{d^2 y}{dx^2}$ by m^2, $\frac{d^3 y}{dx^3}$ by m^3 and so on.

...(1)

...(2)

given

gives

tinct,

) by

Example 1: *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0.$

[B.C.A. (Bhopal) 2012]

Solution: Here, the auxiliary equation is

$$m^2 - 3m - 4 = 0$$

i.e., $(m-4)(m+1) = 0,$ $[\therefore m = -1, 4]$

Hence, the general solution of the given differential is

$$y = c_1 e^{-x} + c_2 e^{4x}$$

where c_1 and c_2 are arbitrary constants.

Example 2: *Solve* $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0.$

[B.C.A. (Agra) 2009]

Solution: Here, the auxiliary equation is

$$m^2 - 4m + 1 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

Hence, the required solution is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x} \text{ or } y = e^{2x}\{c_1 e^{x\sqrt{3}} + c_2 e^{-x\sqrt{3}}\},$$

where c_1 and c_2 are arbitrary constants.

Example 7: Solve $\frac{d^3y}{dx^3} - 8y = 0.$

[B.C.A. (Purvanchal) 2010, 07, 03]

Solution: Here, the auxiliary equation is

$$m^3 - 8 = 0 \Rightarrow (m-2)(m^2 + 2m + 4) = 0$$

$$\Rightarrow \quad m = 2, m = \frac{-2 \pm \sqrt{4-16}}{2} \Rightarrow m = 2, m = -1 \pm i\sqrt{3}.$$

Hence, the required general solution is

$$y = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x).$$

Example 8: Solve $\frac{d^4y}{dx^4} + m^4 y = 0.$

[B.C.A. (Meerut) 2006 (B.P.) 05; B.C. A. (Agra) 2002]

Solution: The given differential equation can be written as

$$(D^4 + m^4)y = 0.$$

Thus, its auxiliary equation is

$$D^4 + m^4 = 0.$$

$$\Rightarrow \quad D^4 + 2D^2m^2 + m^4 - 2D^2m^2 = 0$$

$$\Rightarrow \quad (D^2 + m^2)^2 - (\sqrt{2}\,Dm)^2 = 0$$

$$\Rightarrow \quad (D^2 - \sqrt{2}\,Dm + m^2)(D^2 + \sqrt{2}\,Dm + m^2) = 0$$

$$\Rightarrow \quad D^2 - \sqrt{2}\,Dm + m^2 = 0, D^2 + \sqrt{2}\,Dm + m^2 = 0$$

$$\Rightarrow \quad D = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}, D = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}$$

$$\Rightarrow \quad D = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}, D = -\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}.$$

Hence, the required general solution is

$$y = e^{mx/\sqrt{2}}\left(c_1 \cos\frac{m}{\sqrt{2}}x + c_2 \sin\frac{m}{\sqrt{2}}x\right) + e^{-mx/\sqrt{2}}\left(c_3 \cos\frac{m}{\sqrt{2}}x + c_4 \sin\frac{m}{\sqrt{2}}x\right)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Exa...

10.9 To Find $\dfrac{1}{f(D)}e^{ax}$

When $\qquad f(a) \neq 0$, where $f(D) = (D-a)^n \phi(D)$

Let $\qquad f(D) = (D-a)^n \, \phi(D)$, where $\phi(a) \neq 0$.

Then
$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^n \phi(D)} e^{ax} = \frac{1}{(D-a)^n} \cdot \frac{1}{\phi(D)} e^{ax}$$
$$= \frac{1}{(D-a)^n} \frac{1}{\phi(a)} e^{ax} = \frac{1}{\phi(a)} \cdot \frac{1}{(D-a)^n} e^{ax} \qquad ...(1)$$

$[\because \phi(a) \neq 0]$

Now
$$\frac{1}{D-a} e^{ax} = e^{ax} \int e^{ax} e^{-ax} dx = xe^{ax},$$
$$\frac{1}{(D-a)^2} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{D-a} e^{ax}$$
$$= \frac{1}{D-a}(xe^{ax}) = e^{ax} \int xe^{ax} . e^{-ax} dx = e^{ax} \int x \, dx = \frac{x^2 e^{ax}}{2}$$
$$\frac{1}{(D-a)^3} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{(D-a)^2} e^{ax} = \frac{1}{D-a} \left(\frac{x^2 e^{ax}}{2} \right)$$
$$= e^{ax} \int \frac{x^2 e^{ax}}{2} . e^{-ax} dx = \frac{x^3 e^{ax}}{3!}.$$

Proceeding in a similar manner, one can see that
$$\frac{1}{(D-a)^n} e^{ax} = \frac{x^n e^{ax}}{n!}. \qquad ...(2)$$

Three lines n/A.

$y = c_1 e \quad + c_2 e$

Example 13: *Solve* $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = e^{-3x}$.

[B.C.A. (Bundelkhand) 2007; B.C.A. (Kanpur) 2007]

Solution: Here, the auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$\Rightarrow \quad (m+1)(m+3) = 0 \quad \text{or} \quad m = -1, -3.$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-3x}$$

and

$$\text{P.I.} = \frac{1}{D^2 + 4D + 3} e^{-3x} = \frac{1}{(D+3)(D+1)} e^{-3x}$$

$$= \frac{1}{D+3}\left\{\frac{1}{(-3)+1} e^{-3x}\right\} = -\frac{1}{2}\frac{1}{D+3} e^{-3x}$$

$$= -\frac{1}{2} e^{-3x} \int e^{-3x} e^{3x}\, dx = -\frac{1}{2} e^{-3x} \int 1\, dx = -\frac{1}{2} x\, e^{-3x}.$$

Hence, the required general solution is

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2} x e^{-3x}.$$

Example 14: *Solve* $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2) \ldots$

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$f(-a^2) \ldots, f(-a^2) \neq 0.$

Example 19: *Solve* $(D^2 + 1)y = \cos 2x.$

[B.C.A. (Agra) 2003]

Solution: The auxiliary equation of the differential equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i, \quad i.e., \ 0 \pm i.$$

$\therefore$

$$(\text{C.F.}) = c_1 \cos x + c_2 \sin x.$$

and

$$(\text{P.I.}) = \frac{1}{D^2 + 1} \cos 2x = \frac{1}{(-2^2) + 1} \cos 2x = -\frac{\cos 2x}{3}.$$

Hence, the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

i.e.,

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{3} \cos 2x.$$

Example 20. ... $d^2 y$

Hence, th...

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}x + \frac{5}{36}.$$

Example 27: *Solve* $(D^2 - 4)y = x^2$. **[B.C.A. (Agra) 2010]**

Solution: Here, the auxiliary equation is

$$m^2 - 4 = 0;\ m = \pm 2.$$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$, where c_1 and c_2 are arbitrary constants.

Also

$$\text{P.I.} = \frac{1}{D^2 - 4} x^2 = \frac{1}{-4(1 - \frac{1}{4}D^2)} x^2$$

$$= -\frac{1}{4}(1 - \frac{1}{4}D^2)^{-1} x^2 = -\frac{1}{4}[1 + \frac{1}{4}D^2 + \ldots] x^2$$

$$= -\frac{1}{4}[x^2 + \frac{1}{4}D^2(x^2)], \text{ as all other terms vanish}$$

$$= -\frac{1}{4}(x^2 + \frac{1}{2}).$$

Hence, the complete solution of given equation is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}(x^2 + \frac{1}{2}).$$

Example 28: *Solve* $(D^3 + 3D^2 + 2D)y = x^2$. **[B.C.A. (Agra) 2004, 03]**

Sequences

4.1 Sequence

[B.C.A. (Delhi) 2012, 09, 08; B.C.A. (Kanpur) 2011, 07; B.C.A. (Agra) 2010, 06, 04]

Let S be any non-empty set. A function whose domain is the set N of natural numbers and whose range is a subset of S, is called a **sequence** in the set S.

Or

sequence in a set S is a rule which assigns to each natural number a unique element of S.

.1.1 Real Sequence

sequence whose range is a subset of R is called a **real sequence** or a sequence of real umber. A sequence is denoted by $< s_n >$ or $\{s_n\}$, where s_n is the nth term of sequence.

g., The sequence $< 1, 8, 27, 64, \ldots\ldots n^3 \ldots\ldots >$ or $< n^3 > \forall\, n \in N$.

NOTE:

The sequence can be denoted by Recursion formula.

Let $a_1 = 1, a_{n+1} = 3a_n$, for all $n \geq 1$

Illustration 1: $\left\langle \frac{1}{n} \right\rangle$ is the sequence $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\ldots >$.

Illustration 2: $\left\langle \frac{n}{2n+1} \right\rangle$ is the sequence $< \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots, \frac{n}{2n+1}, \ldots >$

Illustration 3: Let $s_1 = 1,\ s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n, \forall n \geq 1$.

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4.1.2 Range of Sequence

The set of all distinct terms of a sequence is called its **range**. [B.C.A. (Meerut) 2012, 09, 00]

∴ The range of sequence $<s_n>$ = The set $\{s_1, s_2, s_3, \ldots\}$

Illustration 1: The range of sequence $<(-1)^n> = \{-1, 1\}$, a finite set.

Illustration 2: The range of sequence $\left\langle \frac{1}{n+1} \right\rangle = \left\{ \frac{1}{n+1} : n \in N \right\}$ is an infinite set.

4.1.3 Constant Sequence

[B.C.A. (Agra) 2011, 07]

A sequence $<s_n>$ defined by $s_n = a,\ \forall n \in N$ is called a **constant sequence.**

Thus, the sequence $<s_n> = <a, a, a, \ldots>$ is a constant sequence. The range of $<s_n> = a$.

4.1.4 Equality of Two Sequences

Two sequences $<s_n>$ and $<t_n>$ are said to be equal if $s_n = t_n,\ \forall n \in N$.

4.1.5 Operations of Sequences

Let $<s_n>$ and $<t_n>$ be two sequences. Then:

1. Sum of sequences $= <s_n + t_n>$
2. Difference of sequences $= <s_n - t_n>$
3. Product of sequence $= <s_n t_n>$
4. Quotient of sequences $= \left\langle \frac{s_n}{t_n} \right\rangle$
5. Reciprocal sequences of $<s_n> = \left\langle \frac{1}{s_n} \right\rangle$.

4.2 Sub-Sequences

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4.3 Bounded Sequences

[B.C.A. (Meerut) 2003, 02]

4.3.1 Bounded Above

A sequence $<s_n>$ is said to be bounded above if the range set of $<s_n>$ is **bounded above** *i.e.*, if there exists a real number (k_1) such that

$$s_n \leq k_1 \, \forall n \in N$$

n

The number k_1 is called an **upper bound** of the sequence $<s_n>$. $(-1)^n$

$y = n^2 \; n \in N$

$\langle 1, 4, 9 \rangle$

4.3.2 Bounded Below

A sequence $<s_n>$ is said to be bounded below if the range set of $<s_n>$ is **bounded below** *i.e.*, if there exists a real number k_2 such that

$$s_n \geq k_2, \, \forall n \in N$$

The number k_2 is called a **lower bound** of the sequence $<s_n>$.

4.3.3 Bounded Sequence

[B.C.A. (Meerut) 2012, 08; B.C.A. (Lucknow) 2010, 04]

A sequence $<s_n>$ is said to be bounded if the range set of $<s_n>$ is both bounded above and bounded below *i.e.*, if there exists two real numbers k_1 and k_2 such that

$$k_2 \leq s_n \leq k_1, \; \forall n \in N$$

Or

A sequence $<s_n>$ is bounded if and only if there exists a real number $k > 0$ such that

$$|s_n| \leq k, \; \forall n \in N$$

It is not necessary that a sequence be bounded above or bounded below.

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4.4 Unbounded Sequence

[B.C.A. (Purvanchal) 2012, 09, 07; B.C.A. (Meerut) 2011, 03]

A sequence $<s_n>$ is said to be **unbounded** if it is either unbounded below or unbounded above.

4.4.1 Supremum (Sup) or Least Upper Bound (l. u. b.)

[B.C.A. (Bundelkhand) 2010; B.C.A. (Agra) 2009]

The least number say M, if exists, of set of the upper bounds of $<s_n>$ is called the **least upper bound** (l, u, b) or the **supremum** (sup) of the sequence $<s_n>$.

4.4.2 Infimum (inf) or Greatest Lower Bound (g. l. b.)

[B.C.A. (Bundelkhand) 2010; B.C.A. (Agra) 2009

The greatest number say m, if it exists of the set of the lower bounds of $<s_n>$ is called th greatest lower bound $(g.l.b.)$ or the infimum (inf) of the sequence $<s_n>$.

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$s_n \to l$

4.6 Convergent Sequences

[B.C.A. (Meerut) 2012, 09, 07, 02]

A sequence $<s_n>$ is said to converge to a number l, if for any given $\in > 0$ there exists a positive integer m such that

$$|s_n - l| < \in, \forall n \geq m$$

The number l is called the limit of the sequence $<s_n>$. It can be represented as

$$\lim_{n \to \infty} s_n = l \text{ or } \lim s_n = l$$

The positive integer m depends on $\in$.

NOTE:

$$|s_n - l| < \in \Rightarrow l - \in < s_n < l + \in$$
$$\Rightarrow s_n \in]l - \in, l + \in[.$$

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Theorem 3: A sequence cannot converge to more than one limit *i.e.*, the limit of a sequence is unique.

[B.C.A. (Meerut) 2012, 09, 08, 07, 05, 02, 01; B.C.A. (Agra) 2012, 09, 08, 06, 04; B.C.A. (Rohilkhand) 2012, 09; B.C.A. (Kanpur) 2010, 07]

Proof: Let us consider the sequence $<s_n>$ converge to two different number l and l_1.

Since $l \neq l_1$, therefore $|l - l_1| > 0$.

Let $$\epsilon = \frac{1}{2}|l - l_1|, \text{ then } \epsilon > 0$$

Since $<s_n>$ converges to l then for every $\epsilon > 0$ there exists $m_1 \in N$ such that

$$|s_n - 1| < \epsilon, \ \forall n \geq m_1 \qquad \text{...(1)}$$

Again, since $< s_n >$ converges to l_1 then for every $\epsilon > 0$ there exists $m_2 \in N$ such that

$$|s_n - l_1| < \epsilon, \ \forall n \geq m_2 \qquad \text{...(2)}$$

Let $m = \max(m_1, m_2)$ then $n \geq m$ for (1) and (2)

$$\therefore \quad |l - l_1| = |(s_n - l) - (s_n - l_1)| \leq |s_n - l| + |s_n - l_1|$$

$$< \epsilon + \epsilon = |l - l_1|$$

Thus $|l - l_1| < |l - l_1|$ which is not possible so $l = l_1$ *i.e.*, the limit of sequence is unique.

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Hence, s_n

Theorem 6: Every convergent sequence is bounded.

[B.C.A. (Meerut) 2010, 08, 05, 04, 02, 00; B.C.A. (Rohilkhand) 2012, 09, 05, 00; B.C.A. (Agra) 2009]

Or

Show that every convergent sequence is bounded. Is converse true ? Give reasons for your answer. [B.C.A. (Kanpur) 2012, 08, 04, 01; B.C.A. (Meerut) 2008]

Proof: Let $<s_n>$ be a sequence which converges to l.

Let $\epsilon = 1 > 0$, then there exists a positive integer m such

$$|s_n - l| < 1, \forall n \geq m$$

$$\Rightarrow \quad l - 1 < s_n < l + 1, \forall n \geq m$$

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Example 3: *If* $s_n = \left\langle \dfrac{n^2+3n+5}{2n^2+5n+7} \right\rangle$ *then show* $<s_n>$ *converge to* $\dfrac{1}{2}$.

[B.C.A. (Rohilkhand) 2011, 08, 05, 02]

Solution: We know l is the limit of s_n if $\lim\limits_{n\to\infty} s_n = l$.

$$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{n^2+3n+5}{2n^2+5n+7}\right\}$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{3}{n}+\frac{5}{n^2}\right)}{2n^2\left(1+\frac{5}{2n}+\frac{7}{2n^2}\right)}$$

$$= \frac{1+0+0}{2(1+0+0)}$$

$$= \frac{1}{2}.$$

Example 4: ***Prove that the sequence*** $<s_n>$ ***where*** $s_n = \dfrac{n}{n^2+1}$ ***is convergent.***

[B.C.A. (Rohtak) 2012]

Solution: The sequence converge to l if $\lim\limits_{n\to\infty} s_n = l$.

$$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{n}{n^2+1}\right\} = \lim_{n\to\infty}\left\{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n\left(1+\frac{1}{n^2}\right)}\right\}$$

$$= 0.$$

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Example 10: Show that the sequence $<s_n>$ *defined by* $s_n = \sqrt{n+1} - \sqrt{n}$, $\forall n \in N$ *convergent.*

[B.C.A. (Rohtak) 2010, 08, 04; B.C.A. (Meerut) 2009, 07, 06]

Solution: We have

$$s_n = \sqrt{n+1} - \sqrt{n}$$

$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\therefore \quad \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{(\sqrt{n+1} + \sqrt{n})} = \frac{1}{\infty} = 0$$

Hence, $<s_n>$ converge to zero.

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Example 11: *Show that Lim $\sqrt[n]{n} = 1$.*

[B.C.A. (Lucknow) 2010, 08; B.C.A. (Kanpur) 2009, 06, 05]

Solution: We have $s_n = \sqrt[n]{n} = (n)^{1/n}$

Taking log on both sides

$$\log s_n = \frac{1}{n} \log n$$

$$\log s_n = \frac{\log n}{n}$$

Taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} \log(s_n) = \lim_{n\to\infty} \frac{\log n}{n}, \qquad \left(\frac{\infty}{\infty}\right) \text{forms}$$

Then applying L Hospital rule

$$\therefore \quad \lim_{n\to\infty} \log(s_n) = \lim_{n\to\infty} \left(\frac{1/n}{1}\right) = \frac{1}{\infty} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} s_n = e^0$$

$$\Rightarrow \quad \lim_{n\to\infty} s_n = 1$$

$$\Rightarrow \quad \lim_{n\to\infty} (n)^{1/n} = 1.$$

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Theorem 9: (Cauchy's First Theorem on Limits)

If $\lim_{n\to\infty} s_n = l$, then $\lim_{n\to\infty} \frac{s_1 + s_2 + \ldots + s_n}{n} = l$.

Proof: Define a sequence $<t_n>$ such that

$$s_n = l + t_n, \ \forall n \in N$$

$\therefore$ $$\lim t_n = 0$$

And $$\frac{s_1 + s_2 + \ldots + s_n}{n} = l + \frac{t_1 + t_2 + \ldots + t_n}{n} \quad \ldots(1)$$

In order to prove the theorem we have to show that $\lim_{n\to\infty} \frac{t_1 + t_2 + \ldots + t_n}{n} = 0$.

Let $\in > 0$ be given, since $\lim t_n = 0$, therefore, these exists a positive integer m, such that

$$|t_n - 0| = |t_n| < \frac{\in}{2} \quad \forall n \geq m \quad \ldots(2$$

Also, since every convergent sequence is bounded, hence there exists a real number $k >$ such that

$$|t_n| \leq k \ \forall n \geq \in N \quad \ldots($$

$\therefore$ $$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| = \left|\frac{t_1 + t_2 + \ldots + t_m}{n}\right| + \left|\frac{t_{m+1} + t_{m+2} + \ldots + t_n}{n}\right|$$

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Example 32: *Prove that* $\lim\left[\frac{(n!)^{1/n}}{n}\right]=\frac{1}{e}$. [B.C.A. (Rohtak) 2008; B.C.A. (Meerut) 2002]

Solution: Let $s_n=\frac{n!}{n^n}$. Than $s_n>0\ \forall n\in N$

we know that if $s_n>0,\ \forall n\in N$, then $\lim (s_n)^{1/n}=\lim\left(\frac{s_{n+1}}{s_n}\right)$

We have $$\left(\frac{s_{n+1}}{s_n}\right)=\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{n^n}{(n+1)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim\left(\frac{s_{n+1}}{s_n}\right)=\lim\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$$

Then $$\lim (s_n)^{1/n}=\lim\left[\frac{(n)!^{1/n}}{n}\right]=\frac{1}{e}.$$

M. Imp

Example 33: *Prove that* $\lim\limits_{n\to\infty}\left[\frac{(3n)!}{(n!)^3}\right]^{1/n}=27$.

[B.C.A. (Lucknow) 2010; B.C.A. (Kanpur) 2008; B.C.A. (Delhi) 2008; B.C.A. (Garhwal) 2007, 04]

Solution: Let $s_n=\frac{(3n)!}{(n!)^3}$. Then $s_n>0\quad\forall n\in N$,

$\lim\limits_{n\to\infty} s_n = l$

Now we know that if $s_n>0\quad\forall n\in N$, then

$$\lim\ (s_n)^{1/n}=\lim\left(\frac{s_{n+1}}{s_n}\right)$$

We have $$\frac{s_{n+1}}{s_n}=\frac{(3n+3)!}{((n+1)!)^3}\cdot\frac{(n)!^3}{(3n)!}$$

3n!

$s_n = 3(n+1)$

$3n+3$

$$=\frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$

$$=\frac{\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^3}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{s_{n+1}}{s_n}\right)=\frac{(3+0)(3+0)(3+0)}{(1+0)^3}=27$$

$\therefore$ $$\lim\ (s_n)^{1/n}=\lim\left[\frac{(3n)!}{(n!)^3}\right]^{1/n}=27.$$

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4.9 Oscillatory Sequences

[B.C.A. (Meerut) 2000]

A sequence $< s_n >$ is said to be an **oscillatory sequence** if it is neither convergent nor divergent.

Illustration 1: The sequence $< (-1)^n >$ oscillates finitely.

Illustration 2: The sequence $< (-1)^n n >$ oscillates infinitely. [B.C.A. (Meerut) 2000]

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4.10 Monotonic Sequence

[B.C.A. (Meerut) 2007]

4.10.1 Definition

1. A sequence $< s_n >$ is said to be **monotonically increasing** or non-decreasing, if $s_n \le s_{n+1}$ for all n *i.e.,* $s_n \le s_m$ for all $n < m$.

2. A sequence $< s_n >$ is said to be **monotonically strictly increasing** if $s_n < s_{n+1}$ $\forall n \in N$.

3. A sequence $< s_n >$ is said to be **monotonically decreasing** or non-increasing, if $s_n \ge s_{n+1}$, $\forall n$

   i.e., $s_n \ge s_m$, $\forall n < m$.

4. A sequence $< s_n >$ is said to be **strictly decreasing** if $s_n > s_{n+1}$, $\forall n \in N$.

5. A sequence $< s_n >$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

For Example: 2/3/3

(i) The sequence $< 1, 2, 3, \ldots\ldots n, \ldots >$ is strictly increasing.

(ii) The sequence $< 2, 2, 4, 4, 6, 6, \ldots >$ is monotonically increasing.

(iii) The sequence $< -\frac{1}{n} >$ is strictly increasing.

(iv) The sequence $< \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots >$ is strictly decreasing.

(v) The sequence $< -2, 2, -4, 4, -6, 6 \ldots >$ is not monotonic.

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Example 37: ***Find the bounds of the sequence*** $< s_n >$

where $$s_n = \frac{4n-1}{5n+2}.$$

[B.C.A. (Garhwal) 2006; B.C.A. (Agra) 2005; B.C.A. (Meerut) 2003]

Solution: We have $$s_n = \frac{4n-1}{5n+2}$$ ①

$n = n+1$

and $$s_{n+1} = \frac{4n+3}{5n+7}$$ ②

$4(n+1)$

$4n+4-1$

$$\therefore \quad s_{n+1} - s_n = \frac{4n+3}{5n+7} - \frac{4n-1}{5n+2}$$

$$= \frac{(4n+3)(5n+2)-(4n-1)(5n+7)}{(5n+7)(5n+2)}$$

$$= \frac{20n^2+8n+15n+6-20n^2-28n+5n+7}{(5n+7)(5n+2)}$$

$$= \frac{13}{(5n+7)(5n+2)} > 0$$

Therefore, the sequence is monotonic increase the lower bound is the first term (for $n = 1$) *i.e.*, $3/7$

Now $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{4n-1}{5n+2} = \lim_{n\to\infty} \frac{4n\left(1-\frac{1}{4n}\right)}{5n\left(1+\frac{2}{5n}\right)}.$$

$$= \frac{4}{5}$$

$\Rightarrow$ $\frac{4}{5}$ is upper bound

These are respectively the greatest lower bound and the least upper bound of the sequence $< s_n > \to 4/5$.

Example 38: ***Prove that the sequence*** $< s_n >$***, where***

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4.12 Cauchy's Sequences

A sequence $<s_n>$ is said to be a **Cauchy's sequence** if given $\in > 0$ there exists $m \in N$ such that

$$|s_n - s_m| < \in, \ \forall n \geq m$$

or

$$|s_{n+p} - s_n| < \in, \ \forall n \geq m \text{ and every } p > 0$$

or

$$|s_p - s_q| < \in, \ \forall p, q \geq m.$$

[B.C.A. (Meerut) 2007, 03]

Example 44: *The sequence* $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots \right\rangle$ *is a cauchy's sequence.*

Solution: Let the given sequence $<s_n>$, where $s_n = \frac{1}{n}$ and $\in > 0$. If $n \geq m$, then

$$|s_n - s_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| = \frac{n-m}{mn} = \frac{n-m}{n} \cdot \frac{1}{m} < \frac{1}{m}$$

$$\left[0 \leq n - m < n \Rightarrow 0 \leq \frac{n-m}{n} < 1\right]$$

If we consider $\frac{1}{m} < \in$, then

$$|s_n - s_m| = \left|\frac{1}{n} - \frac{1}{m}\right| < \in \ \ \forall n \geq m$$

Hence, the given sequence is a Cauchy's sequence.

Theorem 25: If $<s_n>$ is a Cauchy's sequence, then $<s_n>$ is bounded.

[B.C.A. (Rohilkhand) 2012, 08, 00; B.C.A. (Agra) 2012, 09; B.C.A. (Kanpur) 2010, 05; B.C.A. (Meerut) 2009, 03, 00]

Proof: Let $<s_n>$ be Cauchy's sequence. Then for every $\in > 0$ *i.e.*, let $\in = 1$, there exists $m \in N$ such that $|s_n - s_m| < 1, \ \forall n \geq m.$

$\Rightarrow$ $(s_m - 1) < s_n < (s_m + 1), \ \forall \ n \geq m$

Let $k_1 = \min \{s_1, s_2, s_3, \ldots\ldots s_{m-1}, s_m - 1\}$

$k_2 = \max \{s_1, s_2, s_3, \ldots\ldots s_{m-1}, s_m + 1\}$

$\therefore$ $k_1 \leq s_n \leq k_2, \ \ \forall n \in N$

Hence, $<s_n>$ is bounded.

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Infinite Series

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5.1 Definition

Series: *"An expression of the form $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots.$ in which every term is followed by another term by some definite rule is called series".*

For example:

1. $1+3+5+7+\ldots\ldots$ is a series.
2. $2+2^2+2^3+\ldots\ldots.$ is a series.
3. $1+2+7+8+11+12+\ldots\ldots\ldots$ is not a series.

Because every term is not followed by another term by a definite rule.

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5.2 Finite Series

A series in which numbers of terms are finite is called **finite series**. It may be expressed as

$$u_1 + u_2 + u_3 + \ldots\ldots + u_n \quad \text{or} \quad \sum_{r=1}^{n} u_r.$$

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5.3 Infinite Series

A series in which numbers of terms are infinite is called **infinite series**. It may be expressed as $u_1 + u_2 + \ldots\ldots\ldots + u_n + \ldots\ldots\ldots$ or $\sum_{n=1}^{\infty} u_n$ or simply by $\sum u_n$, where u_n is called **general terms** or nth term of the sequence.

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5.3.2 Sequence of Partial Sums

If $\sum u_n = u_1 + u_2 + u_3 + u_4 + \ldots$ is an infinite series. Now consider the sums

$$S_1 = u_1$$

$$S_2 = u_1 + u_2$$

$$S_3 = u_1 + u_2 + u_3$$

$$\vdots \qquad \vdots$$

$$S_n = u_1 + u_2 + \ldots\ldots + u_n$$

The sequence $< S_n > = < S_1, S_2, S_3, \ldots\ldots\ldots S_n, \ldots\ldots >$ is called **sequence of partial sums**.

5.3.3 Behaviour of an Infinite Series in the Form of Partial Sums

1. **Convergent Series:** The series Σu_n is said to be convergent, if the sequence $< S_n >$ of partial sums of Σu_n is convergent.
2. **Divergent Series:** The series Σu_n is said to be divergent, if the sequence $< S_n >$ of partial sums of Σu_n is divergent.
3. **Oscillatory Series:** The series Σu_n is said to oscillate finitely or infinitely, if the sequence $< S_n >$ of partial sums of Σu_n oscillates finitely or infinitely.

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Example 3: ***Show that the series*** $\Sigma(-1)^{n-1}2$ ***oscillates.***

$\Sigma(-1)^{n-1} \cdot 2$

Solution: $\Sigma(-1)^{n-1}2 = 2-2+2-2+2-\ldots..$

$u_1 = 2$

$$S_1 = 2, S_2 = 0, S_3 = 2, S_4 = 0\ldots$$

$u_2 = -2$

Sequence of partial sums $<S_n> = <S_1, S_2, S_3, \ldots..> = <2,0,2,0\ldots\ldots..>$ is an oscillatory sequence and oscillate finitely.

$u_3 =$

Hence, the series Σu_n oscillates finitely.

Example 4: ***Show that the series*** $\Sigma\, u_n = \Sigma(-1)^n n$ ***is not convergent.***

$S_1 = u_1$

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Solution: $\Sigma(-1)^n n = -1+2-3+4-5+\ldots\ldots$

$$S_1 = 1, S_2 = -1+2 = 1, S_3 = -1+2-3 = -2$$

$S_2 = u_1 +$

$$S_4 = -1+2-3+4 = 2, S_5 = -1+2-3+4-5 = -3\ \ldots\ldots..$$

Sequence of partial sums $= <S_n> = <S_1, S_2, S_3, S_4, S_5, \ldots..>$

$S_3 = u$

$$= <-1, 1, -2, 2, -3, \ldots..> \text{ is not convergent}$$

and therefore, Σu_n is not convergent.

Example 5 ***Show that the series*** $\Sigma u_n = \Sigma \sin\dfrac{n\pi}{3}$ ***is not convergent.***

Solution: $$\Sigma\, u_n = \Sigma \sin\frac{n\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 0 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \ldots$$

Sequence of partial sums $= <S_n> = <S_1, S_2, S_3\ldots\ldots.>$

$$= <\frac{\sqrt{3}}{2}, \sqrt{3}, \sqrt{3}, \frac{\sqrt{3}}{2}, 0, 0, \frac{\sqrt{3}}{2}, \sqrt{3}\ldots..>$$

$S_1 = u_1 =$

$$\overline{\lim_{n\to\infty}}\, S_n = \sqrt{3} \text{ and } \underline{\lim_{n\to\infty}}\, S_n = 0$$

$S_2 = u_1 + u$

$2 - 2 =$

i.e., $$\overline{\lim_{n\to\infty}}\, S_n \neq \underline{\lim_{n\to\infty}}\, S_n$$

Therefore, $<S_n>$ is not convergent and hence $\Sigma\, u_n$ is not convergent.

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Remember:

1. Σu_n convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$.
2. $\lim_{n\to\infty} u_n = 0 \Rightarrow \Sigma u_n$ may or may not be convergent.
3. $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \Sigma u_n$ is not convergent.

Example 6: ***Discuss the convergence of the series***

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$$\Sigma u_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots .$$

Solution: $\Sigma u_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots..$

is a geometric series with common ratio $\frac{1}{3} < 1$, so Σu_n is convergent and $u_n = \frac{1}{3^n}$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3^n} = \frac{1}{3^\infty} = \frac{1}{\infty} = 0$$

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NOTE:

1. If $\lim_{n\to\infty} u_n = 0$, then we cannot say anything about the behaviour of the series
2. If $\lim_{n\to\infty} u_n \neq 0$, then definitely Σu_n does not converse.

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Example 7: ***Prove that the series $\Sigma\, u_n = \Sigma \dfrac{n}{n+1}$ does not converse.***

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Solution: We have

$$\Sigma\, u_n = \Sigma \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots\ldots$$

$$u_n = \frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}}$$

4

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\dfrac{1}{n}\right)} = 1+0 = 1 \neq 0$$

So, $\Sigma\, u_n$ does not converse.

$1+0 \Rightarrow 1 \neq 0$

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Example 8: ***Prove that the series $\Sigma \sqrt{\dfrac{n}{2(n+1)}}$ is not convergent.***

[B.C.A. (Avadh) 2008, 06]

Solution: Let $\Sigma\, u_n = \Sigma \sqrt{\dfrac{n}{2(n+1)}} = \sqrt{\dfrac{1}{4}} + \sqrt{\dfrac{2}{6}} + \ldots + \sqrt{\dfrac{n}{2(n+1)}} + \ldots\ldots$

$$u_n = \sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{1}{1+1/n}}\right\}$$

$\dfrac{1}{1+0}$ 1

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{1}{1+1/n}}\right\} = \frac{1}{\sqrt{2}} \neq 0$$

$\lim\limits_{n\to\infty} \dfrac{1}{3n}$

Hence, the given series does not converse.

3

Example 9: ***Show that the series $\Sigma \cos \dfrac{1}{n}$ does not converse.***

Solution: Let $\qquad \Sigma\, u_n = \Sigma \cos \dfrac{1}{n}$

$\Rightarrow \qquad u_n = \cos \dfrac{1}{n}$

$\Rightarrow \qquad \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \cos \dfrac{1}{n} = \cos \dfrac{1}{\infty}$

$$= \cos 0 = 1 \neq 0$$

Hence, $\Sigma \cos \dfrac{1}{n}$ is not convergent.

Example 10: ***Show that the series $\sum^{\infty} \left(\dfrac{1}{n}\right)^{1/n}$ does not converse.***

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Example 14: Test the convergence of the series whose nth term is :

(i) $\dfrac{\sqrt{n+1}-\sqrt{n-1}}{n}$ (ii) $\dfrac{1}{\sqrt{n}+\sqrt{n+1}}$ (iii) $\sqrt{n+1}-\sqrt{n}$

[B.C.A. (Meerut) 2009, 04]

(iv) $\sqrt{n^3+1}-\sqrt{n^3}$ (v) $\sqrt{n^4+1}-\sqrt{n^4}$ (vi) $[(n^3+1)^{1/3}-n]$ [B.C.A. (Meerut) 2008]

(vii) $\sum \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}$.

[B.C.A. (Meerut) 2009, 08]

Solution: (i) We have

$$u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \times \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{(n+1)-(n-1)}{n[\sqrt{n+1}+\sqrt{n-1}]} = \frac{1}{n^{3/2}}\left[\frac{2}{\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}\right]$$

Taking $v_n = \dfrac{1}{n^{3/2}}$, then $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}$ is a convergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{2}{\sqrt{\left(1+\frac{1}{n}\right)}+\sqrt{\left(1-\frac{1}{n}\right)}}\right] = \frac{2}{2} = 1$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

(ii)
$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}}\left\{\frac{1}{1+\sqrt{\left(1+\frac{1}{n}\right)}}\right\}$$

Taking $v_n = \dfrac{1}{\sqrt{n}}$, so that $\Sigma\, v_n = \Sigma \dfrac{1}{n^{1/2}}$ is divergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left\{\frac{1}{1+\sqrt{1+\frac{1}{n}}}\right\} = \frac{1}{2}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is a divergent series.

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5.3.9 Cauchy's Root Test (or Cauchy's Radical Test)

Let Σu_n be a series of positive terms such that

$$\lim_{n \to \infty} (u_n)^{1/n} = l. \text{ Then,}$$

1. Σu_n converges if $l < 1$
2. Σu_n diverges if $l > 1$
3. Test fails $l = 1$.

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Example 25: Test the convergence of the following series :

[B.C.A. (Meerut) 2011, 04, 03, 02, 00]

(i) $\Sigma\left(\frac{n}{n+1}\right)^{n^2}$ or $\Sigma\left(1+\frac{1}{n}\right)^{-n^2}$ (ii) $\Sigma\frac{x^n}{n^n}$ [B.C.A. (I.G.N.O.U.) 2010]

(iii) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$ (iv) $\Sigma\, x^n n^n (x>0)$

(v) $\Sigma\left(\frac{n+1}{3n}\right)^n$ (vi) $\Sigma\frac{n^{n^2}}{(n+1)^{n^2}}$

[B.C.A. (Bundelkhand) 2008]

(vii) $\Sigma(n^{1/n}-1)^n$ (viii) $\Sigma\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ [B.C.A. (Kanpur) 2009]

(ix) $\sum_{n=1}^{\infty}\frac{n^3}{3^n}$ (x) $\Sigma\frac{1}{n^n}$ [B.C.A. (Purvanchal) 2011]

(xi) $\Sigma 2^{-n-(-1)^n}$.

Solution: (i) Let $u_n=\left(\frac{n}{n+1}\right)^{n^2}$, then $(u_n)^{1/n}=\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n}$

$$=\left(\frac{n}{n+1}\right)^n=\left(\frac{n+1}{n}\right)^{-n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1$$

By Cauchy's root test, the given series $\Sigma\, u_n$ is convergent.

e

$\frac{1}{2}$

$\Rightarrow 0.5<1$

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5.3.10 D' Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n} = l$$

If Σu_n is a series of positive terms such that

$l < 1$ cgt

$$\lim_{n \to \infty} \frac{u_{(n+1)}}{u_n} = l, \text{ then the series}$$

$l > 1$ dgt

cgt

1. Converges, if $l < 1$ $\quad l < 1$ dgt $\qquad$ 2. Diverges, if $l > 1$ and $\quad l > 1$

3. The test fails, if $l = 1$ (*i.e.*, the series may converge, it may diverge if $l = 1$).

Proof: We have $\Sigma u_n = u_1 + u_2 + \ldots + u_{n-1} + u_n + u_{n+1} + \ldots$ and if $\lim \frac{u_{n+1}}{u_n} = l$, then for

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